Introduction to the Phase Plane

We consider the following system of linear differential equations

$$x'(t) = ax(t) + by(t)$$
$$y'(t) = cx(t) + dy(t).$$

We assume that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has a nonzero determinant so that the origin $(x, y) = (0, 0)$ is an isolated critical point of the system. To denote a solution of the system, we will use the notation

$$\mathbf{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

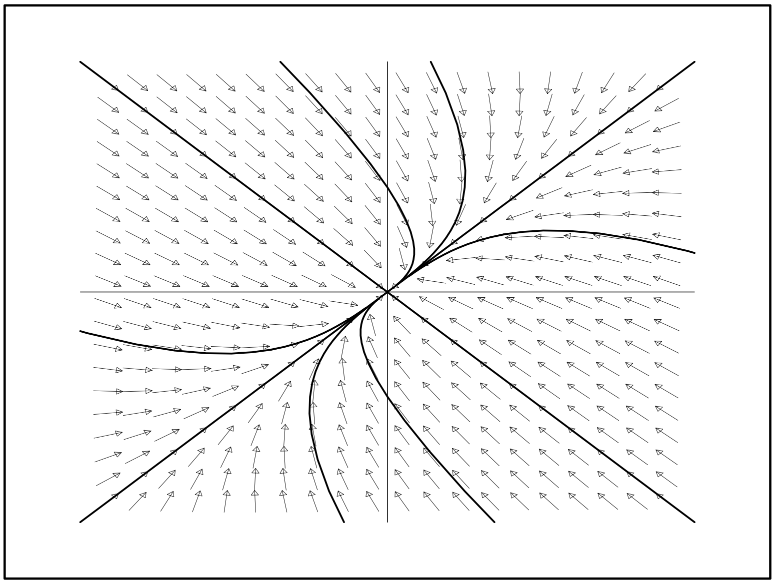
The behavior of the solutions of the system near the origin depends on the nature of the eigenvalues λ_1 and λ_2 of the matrix A. We will discuss each of the following cases

- Real and distinct eigenvalues of the same sign,
- Real eigenvalues of opposite sign,
- Real and equal eigenvalues,
- Complex conjugates eigenvalues with nonzero real part,
- Pure imaginary eigenvalues.

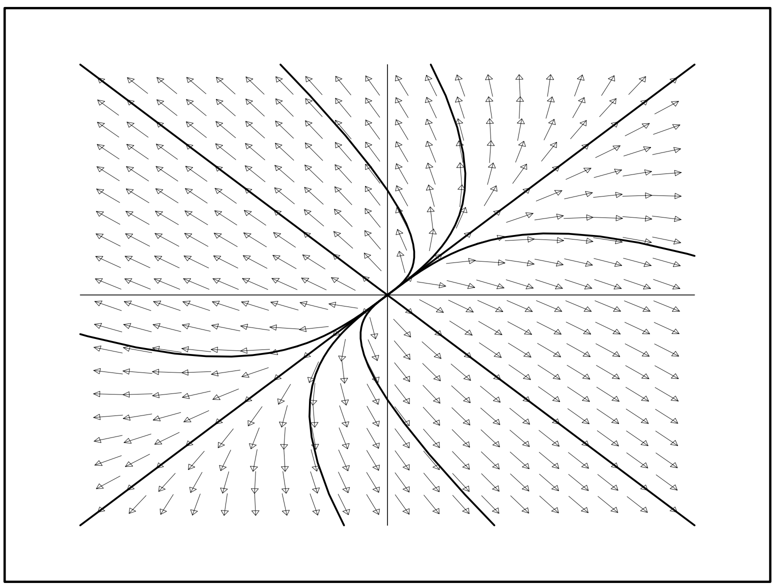
Real and distinct eigenvalues of the same sign. In this case the matrix A has two linearly independent eigenvectors $\mathbf{K}_1$ and $\mathbf{K}_2$ and the general solution of the system is

$$\mathbf{X}(t) = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t}.$$

The origin is called an **improper node**. If $\lambda_1, \lambda_2 < 0$, then the solution $\mathbf{X}(t)$ approaches the origin as t increases so that the origin is said to be asymptotically stable. If $\lambda_1, \lambda_2 > 0$, then the origin is unstable since $\mathbf{X}(t)$ moves away from the origin as t increases. The following figures represent typical phase portrait of these two cases.



Asymptotically stable improper node. $\lambda_1, \lambda_2 < 0$

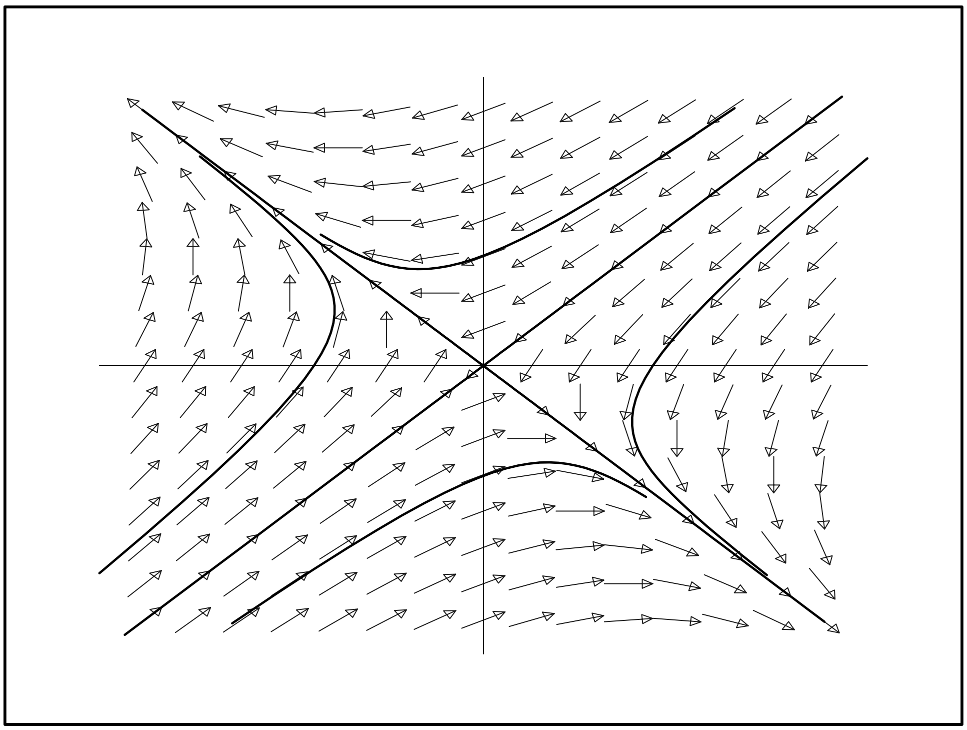


Unstable improper node. $\lambda_1, \lambda_2 > 0$

Real eigenvalues of opposite sign. In this case, the solution is still of the form

$$\mathbf{X}(t) = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t},$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are two linearly independent eigenvectors. If $\lambda_2 < 0 < \lambda_1$, a solution would approach the origin if it started on a point along $\mathbf{K}_2$ and move away from the origin if it started on a point along $\mathbf{K}_1$. The origin in this case is an **unstable saddle point**. The following figure shows a typical phase portrait of this case.



Unstable saddle point.

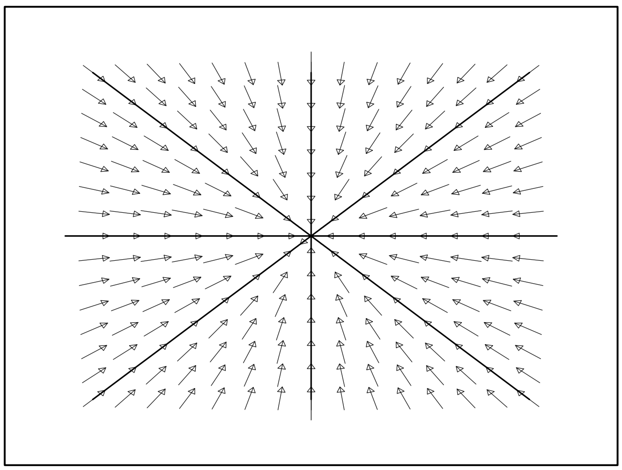
Real and equal eigenvalues. In this case, the situation depends on whether or not the matrix A has two linearly independent eigenvectors $\mathbf{K}_1$ and $\mathbf{K}_2$. If so, then the general solution is of the form

$$\mathbf{X}(t) = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t}$$

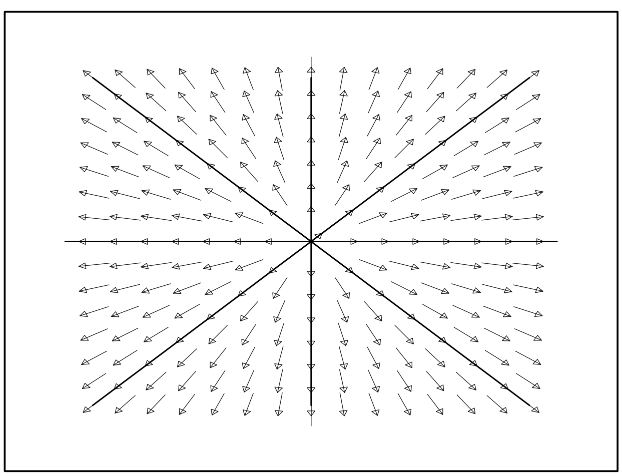
and since $\lambda_1 = \lambda_2 = \lambda$, then

$$\mathbf{X}(t) = (c_1\mathbf{K}_1 + c_2\mathbf{K}_2)e^{\lambda t}.$$

The trajectories in this case are straight lines through the origin. The origin is called a **proper node**. The next two figures represent typical phase portraits of this case.



Asymptotically stable proper node. $\lambda < 0$



Unstable proper node. $\lambda > 0$

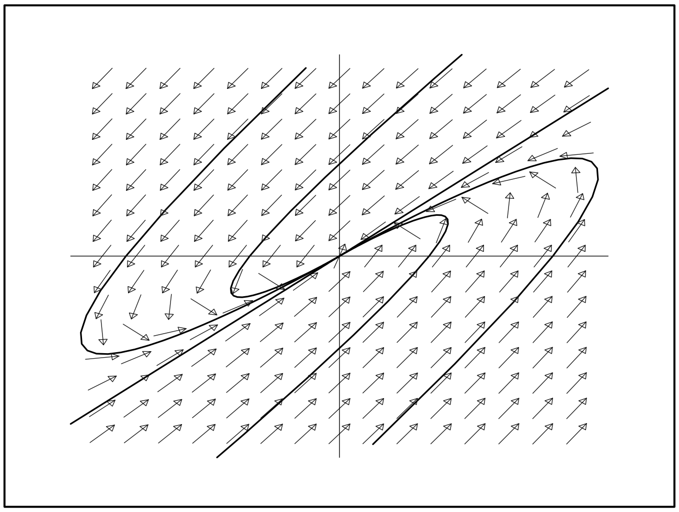
If the matrix A has only one independent eigenvector $\mathbf{K}$, then the general solution is of the form

$$\mathbf{X}(t) = c_1\mathbf{K}e^{\lambda t} + c_2(\mathbf{K}t + \mathbf{P})e^{\lambda t}$$

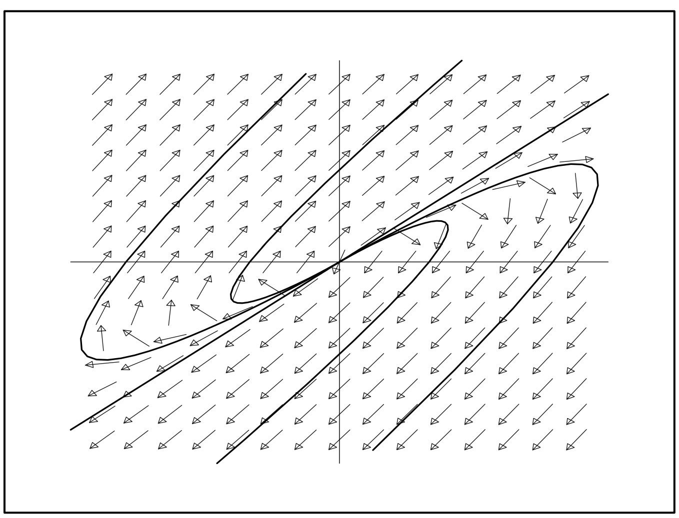
where $\mathbf{P}$ is a generalized eigenvector that satisfies

$$(A - \lambda I)\mathbf{P} = \mathbf{K}.$$

The origin in this case is an **improper node**.



Asymptotically stable improper node. $\lambda < 0$

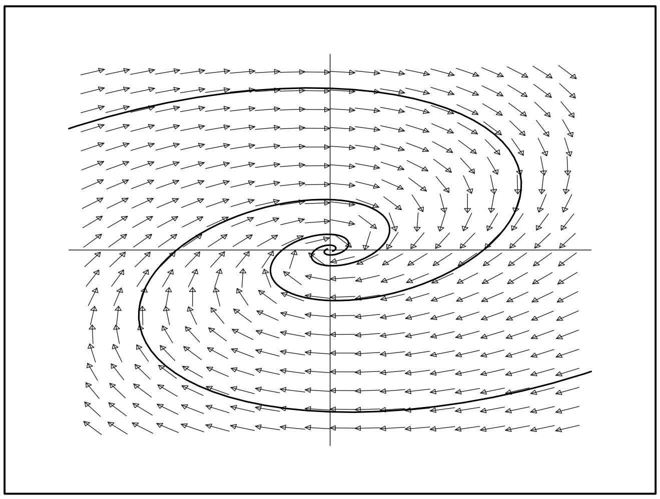


Unstable improper node. $\lambda > 0$

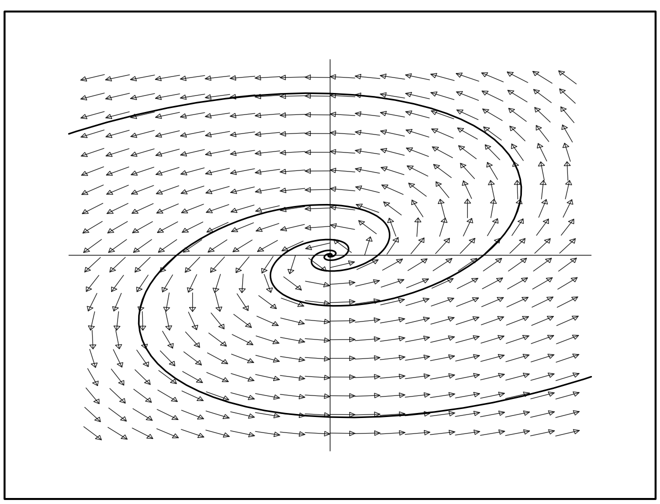
Complex conjugates eigenvalues with nonzero real part. Let's assume that the eigenvalues of the matrix A are $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ with associated eigenvectors $\mathbf{K}_1 = \mathbf{A} + \mathbf{B}i$ and $\mathbf{K}_2 = \mathbf{A} - \mathbf{B}i$. In this case, we have two linearly independent solution

$$\mathbf{X}_1(t) = (\mathbf{A}\cos\beta t - \mathbf{B}\sin\beta t)e^{\alpha t}$$
$$\mathbf{X}_2(t) = (\mathbf{B}\cos\beta t + \mathbf{A}\sin\beta t)e^{\alpha t}.$$

The general solution is $\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t)$. In this case, the origin is a **spiral point**.



Asymptotically stable spiral point. $\alpha < 0$

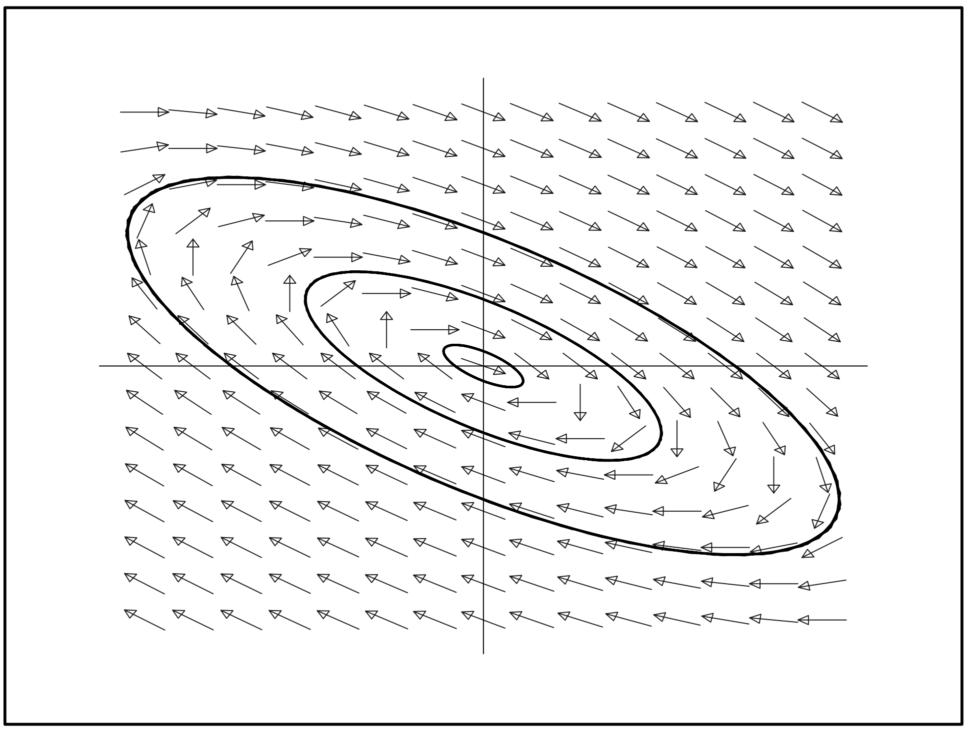


Unstable spiral point. $\alpha > 0$

Pure imaginary eigenvalues. If the eigenvalues of A are $\lambda_1 = \beta i$ and $\lambda_2 = -\beta i$ with associated eigenvectors $\mathbf{K}_1 = \mathbf{A} + \mathbf{B}i$ and $\mathbf{K}_2 = \mathbf{A} - \mathbf{B}i$, then we have two linearly independent solution

$$\mathbf{X}_1(t) = \mathbf{A}\cos\beta t - \mathbf{B}\sin\beta t$$
$$\mathbf{X}_2(t) = \mathbf{B}\cos\beta t + \mathbf{A}\sin\beta t.$$

The general solution is then $\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t)$. In this case, the origin is called a **stable center**. The following figure shows a typical phase portrait of this case.



Stable center.