A proof that e is irrational

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We will prove that the number $e = 2.71828\ldots$ is irrational.

Proof. From Taylor's theorem, we know that for any positive integer n,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n, \quad \text{with } 0 < R_n < \frac{3}{(n+1)!}.$$

Assume that e is rational, i.e., $e = a/b$ for two positive integers a and b. Choose

$$n > \max(b, 3).$$

Then,

$$\frac{a}{b} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n$$

which implies that

$$\frac{n!\,a}{b} - \left(n! + \frac{n!}{1!} + \frac{n!}{2!} + \cdots + \frac{n!}{n!}\right) = n!\,R_n. \tag{1}$$

Since $n > b$, then $n!a/b$ is an integer. Therefore $n!\,R_n$ is also an integer since all terms on the left of equation (1) are integers.

Since $n > 3$ and

$$0 < R_n < \frac{3}{(n+1)!},$$

then

$$0 < n!\,R_n < \frac{3}{n+1} < \frac{3}{4},$$

which is impossible if $n!\,R_n$ is an integer. This gives us a contradiction, therefore e is irrational. QED