Three equivalent definitions of e

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There are several ways to define the number e. Three common definitions are the following.

1. $e = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

2. $e = \displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

3. e is the positive real number such that $\displaystyle\int_1^e \frac{1}{t}\,dt = 1$.

We will show that all three definitions are well-defined† and equivalent.

Let's first consider the sequence

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

It is clear that $\{s_n\}$ is an increasing sequence, i.e., $s_n < s_{n+1}$ for all $n = 0, 1, 2, \ldots$ Starting from $n = 1$, we have $n! = 1 \cdot 2 \cdot 3 \cdots n \geq 1 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1}$, therefore

$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

Since $\{s_n\}$ is increasing and bounded above, then $\displaystyle\lim_{n\to\infty} s_n$ exists.

Now let's consider the sequence

$$t_n = \left(1 + \frac{1}{n}\right)^n.$$

From the binomial theorem, we have

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\ &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right) + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n^3}\right) + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}\left(\frac{1}{n^n}\right) \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

Observe that for all $k = 1, 2, \ldots, n-1$, we have

$$0 < \left(1 - \frac{k}{n}\right) < 1, \quad \text{and} \quad \left(1 - \frac{k}{n}\right) < \left(1 - \frac{k}{n+1}\right).$$

†The term *well-defined* is used for something that is defined in an unambiguous way. For example, if we define a certain number as the limit of a sequence, the definition is well-defined provided that we can show that the limit exists.

It is then clear that for all $n = 1, 2, 3, \ldots$ we have

$$t_n < t_{n+1} \quad \text{and} \quad t_n \leq s_n < 3.$$

Since $\{t_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} t_n$ exists and satisfies

$$\lim_{n\to\infty} t_n \leq \lim_{n\to\infty} s_n. \tag{1}$$

Now let m be a fixed integer such that $2 \leq m \leq n$, then

$$1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right) \leq t_n.$$

By taking the limit as $n \to \infty$ we deduce that

$$s_m = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \leq \lim_{n\to\infty} t_n.$$

By taking the limit as $m \to \infty$ we get

$$\lim_{m\to\infty} s_m \leq \lim_{n\to\infty} t_n. \tag{2}$$

Combining inequalities (1) and (2), we deduce that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n.$$

This establishes the equivalence of definitions 1 and 2.

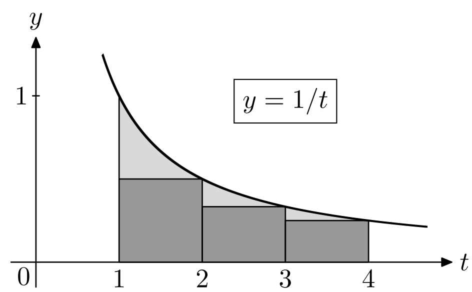
Now let's consider the function

$$f(x) = \int_1^x \frac{1}{t}\, dt, \quad \text{for } x > 0.$$

Since $1/t$ is positive and continuous over $t > 0$, then $f(x)$ is a strictly increasing differentiable function satisfying

$$f'(x) = \frac{1}{x}.$$

Since $f(1) = 0$ and $f(4) \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$, we see that definition 3 is *well-defined* since we can guarantee the existence of a unique number $1 < e < 4$ such that $\int_1^e \frac{1}{t}\, dt = 1$.



The function $f(x)$ has the property that for any n and any $x > 0$, we have

$$f(x^n) = nf(x). \tag{3}$$

Proof. Let $g(x) = f(x^n)$ and $h(x) = nf(x)$. Since $f(1) = 0$, we have $g(1) = h(1) = 0$. From the chain rule we deduce that

$$g'(x) = f'(x^n) \cdot nx^{n-1} = \frac{1}{x^n} \cdot nx^{n-1} = \frac{n}{x}$$

and

$$h'(x) = nf'(x) = \frac{n}{x}.$$

Since $g'(x) = h'(x)$ for all $x > 0$ and $g(1) = h(1)$, then $g(x) = h(x)$ for all $x > 0$. □

Let

$$a = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Since f is continuous, then

$$f(a) = f\left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right) = \lim_{n\to\infty} f\left(\left(1 + \frac{1}{n}\right)^n\right).$$

From property (3), we get

$$\begin{aligned}
f(a) &= \lim_{n\to\infty} n\, f\left(1 + \frac{1}{n}\right) \\
&= \lim_{n\to\infty} \frac{f\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{f\left(1 + \frac{1}{n}\right) - f(1)}{\frac{1}{n}} \qquad \text{(since } f(1) = 0\text{)} \\
&= \lim_{h\to 0} \frac{f\left(1 + h\right) - f(1)}{h} \\
&= f'(1) \\
f(a) &= 1.
\end{aligned}$$

This establishes the equivalence of definitions 2 and 3. Therefore, all three definitions of e are equivalent.

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Observe that definition 2 is equivalent to

$$\lim_{h\to 0^+} (1 + h)^{1/h} = e.$$

To study $\lim_{h\to 0^-} (1 + h)^{1/h}$, observe that

$$\lim_{h\to 0^-} (1 + h)^{1/h} = \lim_{k\to 0^+} (1 - k)^{-1/k} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{-n}.$$

Now,

$$\begin{aligned}
\left(1 - \frac{1}{n}\right)^{-n} &= \left(\frac{n-1}{n}\right)^{-n} \\
&= \left(\frac{n}{n-1}\right)^{n} \\
&= \left(1 + \frac{1}{n-1}\right)^{n} \\
&= \left(1 + \frac{1}{n-1}\right)^{n-1} \left(1 + \frac{1}{n-1}\right).
\end{aligned}$$

As $n \to \infty$, we have $\left(1 + \frac{1}{n-1}\right)^{n-1} \to e$ and $\left(1 + \frac{1}{n-1}\right) \to 1$. Therefore,

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{-n} = e.$$

We can conclude that

$$\lim_{h\to 0}(1+h)^{1/h} = e.$$

This is equivalent to the important limit

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1.$$

We can use this limit to prove that $\frac{d}{dx}e^x = e^x$.

Proof.

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{(e^h-1)e^x}{h}\\ &= \left(\lim_{h\to 0}\frac{e^h-1}{h}\right)e^x\\ &= 1\cdot e^x\\ &= e^x.\end{aligned}$$

□

Now let $\ln x = \log_e x$ so that

$$e^{\ln x} = x, \quad \text{for all } x > 0.$$

Using the chain rule we deduce that

$$e^{\ln x}\frac{d}{dx}\ln x = 1 \implies \frac{d}{dx}\ln x = \frac{1}{x}.$$

We can use the formula in definition 1 to numerically estimate e. For example by computing

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{100!}$$

we obtain

$$e \approx 2.718\,281\,828\,459\,045.$$

To learn more about the number e from a historical point of view, I recommend the wonderful book [1].

References

[1] Eli Maor, *e: The Story of a Number*, Princeton University Press, (1994)