DIRICHLET'S TEST FOR CONVERGENCE OF A SERIES

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It is easy to see that the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^p}$ is absolutely convergent for $p > 1$. We will show that the series is conditionally convergent for $0 < p \leq 1$. Our proof is based on the following theorem.

Dirichlet's Theorem. *Let a_n and b_n be two sequences. If there exists a number M such that*

$$\left| \sum_{k=1}^{n} a_k \right| \leq M \text{ for all } n = 1, 2, 3, \ldots$$

and if the sequence b_n satisfies

$$0 \leq b_{n+1} \leq b_n \text{ and } \lim_{n \to \infty} b_n = 0$$

then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Observe that if $a_n = (-1)^{n+1}$, we get Leibnitz's test for the convergence of the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$.

Recall that every Cauchy sequence of real numbers is convergent, i.e., a sequence a_n converges if for every $\varepsilon > 0$, there is an N such that $|a_n - a_m| < \varepsilon$ for all $n, m > N$ (for a proof, see [1]). We will prove Dirichlet's Theorem by showing that the sequence $r_n = \sum_{k=1}^{n} a_k b_k$ is a Cauchy sequence. For this, we will use the following Lemma.

Abel's Lemma. *Let a_n and b_n be two sequences and let $s_n = \sum_{k=1}^{n} a_k$. Then*

$$\sum_{k=1}^{n} a_k b_k = s_n b_{n+1} + \sum_{k=1}^{n} s_k (b_k - b_{k+1}).$$

Proof. Note that $a_n = s_n - s_{n-1}$, where $s_0 = 0$. Then

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (s_k - s_{k-1}) b_k = \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n} s_{k-1} b_k$$

and

$$\sum_{k=1}^{n} s_{k-1} b_k = \sum_{k=1}^{n} s_k b_{k+1} - s_n b_{n+1}.$$

This completes the proof. □

Proof of Dirichlet's Theorem. Let

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad r_n = \sum_{k=1}^{n} a_k b_k.$$

From Abel's Lemma we get

$$r_n - r_m = s_n b_{n+1} - s_m b_{m+1} + \sum_{k=m+1}^{n} s_k (b_k - b_{k+1}).$$

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Since $0 \le b_{k+1} \le b_k$, and $|s_n| \le M$, we get

$$\begin{aligned} |r_n - r_m| &\le M(b_{n+1} + b_{m+1}) + M \sum_{k=m+1}^{n} (b_k - b_{k+1}) \\ &= M(b_{n+1} + b_{m+1} + b_{m+1} - b_{n+1}) \\ &= 2Mb_{m+1}. \end{aligned}$$

Now, let $\varepsilon > 0$. Since $b_n \to 0$ as $n \to \infty$, we can choose N so that $m > N$ implies $b_{m+1} < \varepsilon/(2M)$. Then $m, n > N$ implies $|r_n - r_m| < \varepsilon$, which completes the proof. □

To show that $\sum_{n=1}^{\infty} \frac{\sin n}{n^p}$ converges for all $p > 0$, we will first find a number M such that

$$\left| \sum_{k=1}^{n} \sin k \right| \le M, \quad \text{for all } n = 1, 2, 3, \ldots$$

To do this we use the identity

$$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta).$$

We then have

$$\sum_{k=1}^{n} 2 \sin k \sin \tfrac{1}{2} = \sum_{k=1}^{n} (\cos(k - \tfrac{1}{2}) - \cos(k + \tfrac{1}{2})) = \cos \tfrac{1}{2} - \cos(n + \tfrac{1}{2}).$$

Since $|\cos \frac{1}{2} - \cos(n + \frac{1}{2})| \le 2$, we get

$$\left| \sum_{k=1}^{n} \sin k \right| \le \frac{1}{\sin \frac{1}{2}}, \quad \text{for all } n = 1, 2, 3, \ldots$$

Since the sequence $b_n = 1/n^p$ satisfies

$$0 \le b_{n+1} \le b_n \text{ and } \lim_{n \to \infty} b_n = 0$$

for all $p > 0$, we conclude that $\sum_{n=1}^{\infty} \frac{\sin n}{n^p}$ converges.

Note that the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^p}$ is also convergent for all $p > 0$. The proof is similar to the one above except that we use the identity

$$2 \cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta).$$

Let's now show that $\sum_{n=1}^{\infty} \frac{\sin n}{n^p}$ is not absolutely convergent for $0 < p \le 1$. It is sufficient to show that the series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n}$ diverges since for $0 < p < 1$

$$0 \le \frac{|\sin n|}{n} \le \frac{|\sin n|}{n^p} \quad \text{for all } n = 1, 2, 3, \ldots$$

Let

$$\alpha_n = \frac{|\sin n| - |\sin(n-1)|}{2n} \quad \text{and} \quad \beta_n = \frac{|\sin n| + |\sin(n-1)|}{2n}.$$

It is easy to see that Dirichlet's Theorem implies that $\sum_{n=1}^{\infty} \alpha_n$ converges. It follows from the divergence of the harmonic series and the inequality

$$|\sin x| + |\sin(x-1)| \ge \sin 1 > 0, \quad \text{for all } x$$

that $\sum_{n=1}^{\infty} \beta_n$ diverges. Since $\frac{|\sin n|}{n} = \alpha_n + \beta_n$, we conclude that $\sum_{n=1}^{\infty} \frac{|\sin n|}{n}$ diverges.

References

[1] Marsden, Jerrold, *Elementary Classical Analysis*, Freeman, (1974)