Laplace Transforms

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Contents

Chapter 1

Problems

1.1 Laplace Transforms

1. Find the Laplace transform of the following functions.

(a) $f(t) = 4t^2 - 2t + 3$

(b) $f(t) = 3\sin 5t - 2\cos 3t$

(c) $f(t) = 3e^{2t} + 5e^{-3t}$

(d) $f(t) = e^{-2t}\,(4\cos 5t + 3\sin 5t)$

(e) $f(t) = t^3e^{2t} + 2te^{-t}$

(f) $f(t) = (1 + e^{3t})^2$

2. Use an appropriate trigonometric identity to find the following Laplace transforms. See page 43 for a list of trigonometric identities.

(a) $\mathscr{L}\{\cos^2 t\}$

(b) $\mathscr{L}\{\sin 2t\cos 2t\}$

(c) $\mathscr{L}\{\sin 3t\cos 4t\}$

(d) $\mathscr{L}\{\sin(\omega t + \phi)\}$

(e) $\mathscr{L}\{\cos(\omega t + \phi)\}$

(f) $\mathscr{L}\{e^{-2t}\sin(3t + \frac{\pi}{6})\}$

3. The hyperbolic sine and hyperbolic cosine are defined by

$$\sinh t = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad \cosh t = \frac{e^t + e^{-t}}{2}.$$

Evaluate both $\mathscr{L}\{\sinh \omega t\}$ and $\mathscr{L}\{\cosh \omega t\}$.

4. Find both $\mathscr{L}\{\cos \omega t\}$ and $\mathscr{L}\{\sin \omega t\}$ by starting from Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

and

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}.$$

5. Use the frequency differentiation property (see page 42) to derive the following formulas.

(a) $\mathscr{L}\{t \sin \omega t\} = \dfrac{2\omega s}{(s^2+\omega^2)^2}$ **(b)** $\mathscr{L}\{t \cos \omega t\} = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

6. Use the frequency shift property (see page 42) to find the following.

(a) $\mathscr{L}\{t\, e^{2t} \sin 3t\}$ **(b)** $\mathscr{L}\{t\, e^{-3t} \cos 2t\}$

1.2 Inverse Laplace Transforms

In problems 1 – 8, find the inverse Laplace transforms of the given functions.

1. $F(s) = \dfrac{1}{s^4}$

2. $F(s) = \dfrac{s+5}{s^2+4}$

3. $F(s) = \dfrac{3s+1}{s^2+5}$

4. $F(s) = \dfrac{1}{(s-3)^5}$

5. $F(s) = \dfrac{s-2}{s^2+2s+5}$

6. $F(s) = \dfrac{s+5}{(s^2+9)^2}$

7. $F(s) = \dfrac{1}{s^2-4s+7}$

8. $F(s) = \dfrac{5s+2}{(s^2+6s+13)^2}$

In problems 9 – 16, use the method of partial fractions to find the given inverse Laplace transforms. See page 45 for a refresher on partial fractions.

9. $\mathscr{L}^{-1}\left\{\dfrac{s+3}{s^2+4s-5}\right\}$

10. $\mathscr{L}^{-1}\left\{\dfrac{s+1}{(2s-1)(s+2)}\right\}$

11. $\mathscr{L}^{-1}\left\{\dfrac{s^2+s+5}{s^4-s^2}\right\}$

12. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+4)}\right\}$

13. $\mathscr{L}^{-1}\left\{\dfrac{s^2+3s+4}{(s-2)^3}\right\}$

14. $\mathscr{L}^{-1}\left\{\dfrac{s^2+s+4}{s^3+9s}\right\}$

15. $\mathscr{L}^{-1}\left\{\dfrac{3s+2}{s^3(s^2+1)}\right\}$

16. $\mathscr{L}^{-1}\left\{\dfrac{2s+1}{(s+3)(s^2+4s+13)}\right\}$

1.3 Initial Value Problems

Use Laplace transforms to solve the following initial-value problems.

1. $y' + 4y = e^t, \quad y(0) = 2$

2. $y' - y = \sin t, \quad y(0) = 1$

3. $y'' + 3y' + 2y = t + 1, \quad y(0) = 1,\ y'(0) = 0$

4. $y'' + 4y' + 13y = 0, \quad y(0) = 1,\ y'(0) = 2$

5. $y'' + 4y = 4t + 8, \quad y(0) = 4,\ y'(0) = -1$

6. $y'' + y' - 2y = 5e^{3t}, \quad y(0) = 1,\ y'(0) = -4$

7. $y'' + y' - 2y = e^t, \quad y(0) = 2,\ y'(0) = 3$

8. $y'' - 2y' + y = e^t, \quad y(0) = 3,\ y'(0) = 4$

9. $y'' + 2y' + 2y = \cos 2t, \quad y(0) = 0,\ y'(0) = 1$

10. $y'' + 4y = \sin 3t, \quad y(0) = 2,\ y'(0) = 1$

11. $y'' + \omega^2 y = \cos \omega t, \quad y(0) = 0,\ y'(0) = 0$

12. $y'' + 2y' + 5y = 3e^{-t}\cos 2t, \quad y(0) = 1, y'(0) = 2$

13. The differential equation for a mass-spring system is

$$mx''(t) + \beta x'(t) + kx(t) = F_e(t).$$

Consider a mass-spring system with a mass $m = 1$ kg that is attached to a spring with constant $k = 5$ N/m. The medium offers a damping force six times the instantaneous velocity, i.e., $\beta = 6$ N· s/m.

(a) Determine the position of the mass $x(t)$ if it is released with initial conditions: $x(0) = 3$ m, $x'(0) = 1$ m/s. There is no external force.

(b) Determine the position of the mass $x(t)$ if it released with the initial conditions: $x(0) = 0$, $x'(0) = 0$ and the system is driven by an external force $F_e(t) = 30\sin 2t$ in newtons with time t measured in seconds.

14. The differential equation for the current $i(t)$ in an LR circuit is

$$L\frac{di}{dt} + Ri = V(t).$$

Find the current in an LR circuit if the initial current is $i(0) = 0\,\text{A}$ given that $L = 2\,\text{H}$, $R = 4\,\Omega$, and $V(t) = 5e^{-t}$ volts with time t measured in seconds..

15. The differential equations for the charge $q(t)$ in an LRC circuit is

$$Lq''(t) + Rq'(t) + \frac{q(t)}{C} = V(t).$$

Find the charge and current in an LRC circuit with $L = 1\,\text{H}$, $R = 2\,\Omega$, $C = 0.25\,\text{F}$ and $V = 50\cos t$ volts if $q(0) = 0\,\text{C}$ and $i(0) = 0\,\text{A}$.

1.4 Step Functions and Impulses

1. Find the Laplace transforms of the following functions.

(a) $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 3, & 1 \le t < 2 \\ -2, & t \ge 2 \end{cases}$

(b) $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

(c) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

(d) $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ t^2, & t \ge 3 \end{cases}$

2. Find the following inverse Laplace transforms.

(a) $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s+3}\right\}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{e^{-2\pi s}}{s^2+4}\right\}$

(c) $\mathscr{L}^{-1}\left\{\dfrac{e^{-s}}{(s-1)(s+2)}\right\}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s(s^2+1)}\right\}$

In problems 3 – 9, use Laplace transforms to solve the initial-value problem.

3. $y' + 2y = f(t), \quad y(0) = 3,$ where $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t \ge 1 \end{cases}$

4. $y'' + y = f(t), \quad y(0) = 1,\ y'(0) = 0,$ where $f(t) = \begin{cases} 0, & 0 \le t \le \pi \\ \sin t, & t \ge \pi \end{cases}$

5. $y'' + y' - 2y = f(t), \quad y(0) = 0,\ y'(0) = 0,$ where $f(t) = \begin{cases} 1, & 0 \le t \le 3 \\ -1, & t \ge 3 \end{cases}$

6. $y'' + 4y = f(t), \quad y(0) = 0,\ y'(0) = 2,$ where $f(t) = \begin{cases} 1, & 0 \le t < \pi, \\ 2, & \pi \le t < 2\pi, \\ 0, & t \ge 2\pi \end{cases}$

7. $y' + 3y = \delta(t-1), \quad y(0) = 2$

8. $y'' + 4y' + 13y = \delta(t-\pi), \quad y(0) = 2,\ y'(0) = 1$

9. $y'' + y = \delta(t - \frac{\pi}{2}) + 2\delta(t-\pi), \quad y(0) = 3,\ y'(0) = -1$

10. Find the Laplace transform of the periodic function $f(t)$ with period $T = 2a$ defined over one period by

$$f(t) = \begin{cases} 1, & 0 \le t < a \\ 0, & a \le t < 2a. \end{cases}$$

11. The differential equation for the current $i(t)$ in an LR circuit is

$$L\frac{di}{dt} + Ri = V(t).$$

Consider an LR circuit with $L = 1\,\mathrm{H}$, $R = 4\,\Omega$ and a voltage source (in volts)

$$V(t) = \begin{cases} 2, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$$

(a) Find the current $i(t)$ if $i(0) = 0$.

(b) Compute the current at $t = 1.5$ seconds, i.e., compute $i(1.5)$.

(c) Evaluate $\lim_{t\to\infty} i(t)$.

(d) Sketch the graph of the current as a function of time.

12. Consider the following function with $a > 0$ and $\epsilon > 0$.

$$\delta_\varepsilon(t-a) = \begin{cases} 0, & 0 \le t < a \\ \dfrac{1}{\varepsilon}, & a \le t < a+\varepsilon \\ 0, & t \ge a+\varepsilon \end{cases}$$

(a) Find $\mathscr{L}\{\delta_\varepsilon(t-a)\}$.

(b) Show that: $\lim_{\varepsilon\to 0} \mathscr{L}\{\delta_\varepsilon(t-a)\} = e^{-as}$.

1.5 Convolution

1. Find the following Laplace transforms.

(a) $\mathscr{L}\{1 * t^4\}$

(b) $\mathscr{L}\left\{\int_0^t \cos\theta\, d\theta\right\}$

(c) $\mathscr{L}\{e^t * \sin t\}$

(d) $\mathscr{L}\left\{\int_0^t \cos\theta \sin(t-\theta)\, d\theta\right\}$

2. Find the following inverse Laplace transforms by using convolution. Do **not** use partial fractions.

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s-1)}\right\}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+1)(s-3)}\right\}$

(c) $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+1))}\right\}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}$

3. Use convolution to derive the following formula.

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} = \frac{\sin\omega t - \omega t\cos\omega t}{2\omega^3}$$

4. Solve for $f(t)$ in the following equation.

$$f(t) = t + e^{2t} + \int_0^t e^{-\theta} f(t-\theta)\, d\theta$$

5. Solve for $f(t)$ in the following equation. Be sure to find *all* solutions.

$$\int_0^t f(\theta) f(t-\theta)\, d\theta = t^3$$

6. Solve for $y(t)$ in the following initial-value problem.

$$y'(t) + \int_0^t e^{-2\theta} y(t-\theta)\, d\theta = 1, \quad y(0) = 0$$

Chapter 2

Solutions

2.1 Laplace Transforms

1. We use the table of Laplace transforms (see page 41) to answer this question.

(a) $\mathscr{L}\{f(t)\} = 4\mathscr{L}\{t^2\} - 2\mathscr{L}\{t\} + 3\mathscr{L}\{1\} = \dfrac{8}{s^3} - \dfrac{2}{s^2} + \dfrac{3}{s}$

(b) $\mathscr{L}\{f(t)\} = 3\mathscr{L}\{\sin 5t\} - 2\mathscr{L}\{\cos 3t\} = \dfrac{15}{s^2+25} - \dfrac{2s}{s^2+9}$

(c) $\mathscr{L}\{f(t)\} = 3\mathscr{L}\{e^{2t}\} + 5\mathscr{L}\{e^{-3t}\} = \dfrac{3}{s-2} + \dfrac{5}{s+3}$

(d) $\mathscr{L}\{f(t)\} = 4\mathscr{L}\{e^{-2t}\cos 5t\} + 3\mathscr{L}\{e^{-2t}\sin 5t\} = \dfrac{4(s+2)}{(s+2)^2+25} + \dfrac{15}{(s+2)^2+25}$

(e) $\mathscr{L}\{f(t)\} = \mathscr{L}\{t^3e^{2t}\} + 2\mathscr{L}\{te^{-t}\} = \dfrac{6}{(s-2)^4} + \dfrac{2}{(s+1)^2}$

(f) Since $f(t) = (1+e^{3t})^2 = 1 + 2e^{3t} + e^{6t}$, we have

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1\} + 2\mathscr{L}\{e^{3t}\} + \mathscr{L}\{e^{6t}\} = \frac{1}{s} + \frac{2}{s-3} + \frac{1}{s-6}.$$

2. **(a)** Since $\cos^2 t = \dfrac{1+\cos 2t}{2}$, we have

$$\begin{aligned} \mathscr{L}\{\cos^2 t\} &= \frac{1}{2}\left(\mathscr{L}\{1\} + \mathscr{L}\{\cos 2t\}\right) \\ &= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right). \end{aligned}$$

(b) Starting from the identity $\sin 2\theta = 2\sin\theta\cos\theta$, if we let $\theta = 2t$ we get

$$\sin 4t = 2\sin 2t\cos 2t \implies \sin 2t\cos 2t = \frac{1}{2}\sin 4t$$

and

$$\mathscr{L}\{\sin 2t\cos 2t\} = \frac{1}{2}\mathscr{L}\{\sin 4t\} = \frac{1}{2}\left(\frac{4}{s^2+16}\right) = \frac{2}{s^2+16}.$$

(c) From the identity $\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$, we deduce

$$\begin{aligned}\sin 3t\cos 4t &= \frac{1}{2}(\sin(3t+4t)+\sin(3t-4t))\\ &= \frac{1}{2}(\sin 7t+\sin(-t))\\ &= \frac{1}{2}(\sin 7t-\sin t)\end{aligned}$$

and

$$\mathscr{L}\{\sin 3t\cos 4t\} = \frac{1}{2}\left(\frac{7}{s^2+49}-\frac{1}{s^2+1}\right).$$

(d) From the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$, we deduce

$$\sin(\omega t+\phi) = \sin\omega t\cos\phi+\cos\omega t\sin\phi$$

and

$$\begin{aligned}\mathscr{L}\{\sin(\omega t+\phi)\} &= \cos\phi\,\mathscr{L}\{\sin\omega t\}+\sin\phi\,\mathscr{L}\{\cos\omega t\}\\ &= \cos\phi\left(\frac{\omega}{s^2+\omega^2}\right)+\sin\phi\left(\frac{s}{s^2+\omega^2}\right).\end{aligned}$$

(e) From the identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta-\sin\alpha\sin\beta$, we deduce

$$\cos(\omega t+\phi) = \cos\omega t\cos\phi-\sin\omega t\sin\phi$$

and

$$\begin{aligned}\mathscr{L}\{\cos(\omega t+\phi)\} &= \cos\phi\,\mathscr{L}\{\cos\omega t\}-\sin\phi\,\mathscr{L}\{\sin\omega t\}\\ &= \cos\phi\left(\frac{s}{s^2+\omega^2}\right)-\sin\phi\left(\frac{\omega}{s^2+\omega^2}\right).\end{aligned}$$

(f) Using the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta$, we deduce

$$\begin{aligned}e^{-2t}\sin(3t+\tfrac{\pi}{6}) &= e^{-2t}(\sin 3t\cos\tfrac{\pi}{6}+\cos 3t\sin\tfrac{\pi}{6})\\ &= e^{-2t}(\tfrac{\sqrt{3}}{2}\sin 3t+\tfrac{1}{2}\cos 3t)\\ &= \tfrac{\sqrt{3}}{2}e^{-2t}\sin 3t+\tfrac{1}{2}e^{-2t}\cos 3t\end{aligned}$$

and

$$\mathscr{L}\{e^{-2t}\sin(3t+\tfrac{\pi}{6})\} = \frac{\sqrt{3}}{2}\left(\frac{3}{(s+2)^2+9}\right)+\frac{1}{2}\left(\frac{s+2}{(s+2)^2+9}\right).$$

3. Using $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$, we deduce

$$\begin{aligned}\mathscr{L}\{\sinh \omega t\} &= \frac{1}{2}\left(\mathscr{L}\left\{e^{\omega t}\right\} - \mathscr{L}\left\{e^{-\omega t}\right\}\right) \\ &= \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right) \\ &= \frac{1}{2}\left(\frac{s+\omega-(s-\omega)}{s^2-\omega^2}\right) \\ &= \frac{\omega}{s^2-\omega^2}\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}\{\cosh \omega t\} &= \frac{1}{2}\left(\mathscr{L}\left\{e^{\omega t}\right\} + \mathscr{L}\left\{e^{-\omega t}\right\}\right) \\ &= \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) \\ &= \frac{1}{2}\left(\frac{s+\omega+s-\omega}{s^2-\omega^2}\right) \\ &= \frac{s}{s^2-\omega^2}.\end{aligned}$$

4. From Euler's formula, we get

$$\mathscr{L}\left\{e^{j\omega t}\right\} = \mathscr{L}\{\cos \omega t\} + j\mathscr{L}\{\sin \omega t\}. \tag{2.1}$$

From $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$, we deduce

$$\mathscr{L}\left\{e^{j\omega t}\right\} = \frac{1}{s-j\omega} = \frac{1}{s-j\omega}\left(\frac{s+j\omega}{s+j\omega}\right) = \frac{s+j\omega}{s^2+\omega^2}. \tag{2.2}$$

Since the real and imaginary parts in equations (2.1) and (2.2) are equal, we conclude that

$$\mathscr{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2} \quad \text{and} \quad \mathscr{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}.$$

5. We will use the property $\mathscr{L}\{tf(t)\} = -F'(s)$.

(a) If $f(t) = \sin \omega t$, we have $F(s) = \dfrac{\omega}{s^2+\omega^2}$. Therefore

$$\begin{aligned}\mathscr{L}\{t \sin \omega t\} &= -F'(s) = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) \\ &= \frac{2\omega s}{(s^2+\omega^2)^2}.\end{aligned}$$

(b) If $f(t) = \cos\omega t$, we have $F(s) = \dfrac{s}{s^2+\omega^2}$. Therefore

$$\begin{aligned}\mathscr{L}\{t\cos\omega t\} &= -F'(s) = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)\\ &= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.\end{aligned}$$

6. We use the frequency shift property $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$.

(a) Since $\mathscr{L}\{t\sin 3t\} = \dfrac{6s}{(s^2+9)^2}$, we conclude that

$$\mathscr{L}\{te^{2t}\sin 3t\} = \frac{6(s-2)}{((s-2)^2+9)^2} = \frac{6(s-2)}{(s^2-4s+13)^2}.$$

(b) Since $\mathscr{L}\{t\cos 2t\} = \dfrac{s^2-4}{(s^2+4)^2}$, we conclude that

$$\mathscr{L}\{te^{-3t}\cos 2t\} = \frac{(s+3)^2-4}{((s+3)^2+4)^2} = \frac{s^2+6s+5}{(s^2+6s+13)^2}.$$

2.2 Inverse Laplace Transforms

1. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4}\right\} = \dfrac{1}{3!}\mathscr{L}^{-1}\left\{\dfrac{3!}{s^4}\right\} = \boxed{\dfrac{t^3}{6}}$

2. $\mathscr{L}^{-1}\left\{\dfrac{s+5}{s^2+4}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\} + \dfrac{5}{2}\mathscr{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = \boxed{\cos 2t + \dfrac{5}{2}\sin 2t}$

3. $\mathscr{L}^{-1}\left\{\dfrac{3s+1}{s^2+5}\right\} = 3\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+5}\right\} + \dfrac{1}{\sqrt{5}}\mathscr{L}^{-1}\left\{\dfrac{\sqrt{5}}{s^2+5}\right\}$

$$\boxed{\mathscr{L}^{-1}\left\{\frac{3s+1}{s^2+5}\right\} = 3\cos\sqrt{5}t + \frac{1}{\sqrt{5}}\sin\sqrt{5}t}$$

4. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-3)^5}\right\} = \dfrac{1}{4!}\mathscr{L}^{-1}\left\{\dfrac{4!}{(s-3)^5}\right\} = \boxed{\dfrac{t^4e^{3t}}{24}}$

5. By completing the square, we get

$$s^2+2s+5 = (s+1)^2+4.$$

Observe that

$$\frac{s-2}{s^2+2s+5} = \frac{(s+1)-3}{(s+1)^2+4}.$$

Therefore,

$$\mathscr{L}^{-1}\left\{\frac{s-2}{s^2+2s+5}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - \frac{3}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$
$$= \boxed{e^{-t}\cos 2t - \frac{3}{2}e^{-t}\sin 2t.}$$

6. $$\mathscr{L}^{-1}\left\{\frac{s+5}{(s^2+9)^2}\right\} = \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{6s}{(s^2+9)^2}\right\} + 5\mathscr{L}^{-1}\left\{\frac{1}{(s^2+9)^2}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{s+5}{(s^2+9)^2}\right\} = \frac{1}{6}t\sin 3t + 5\left(\frac{\sin 3t - 3t\cos 3t}{2(3^3)}\right)$$
$$= \boxed{\frac{1}{6}t\sin 3t + \frac{5}{54}\sin 3t - \frac{5}{18}t\cos 3t.}$$

7. By completing the square, we get

$$s^2-4s+7 = (s-2)^2+3.$$

Therefore,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2-4s+7}\right\} = \frac{1}{\sqrt{3}}\mathscr{L}^{-1}\left\{\frac{\sqrt{3}}{(s-2)^2+3}\right\}$$
$$= \boxed{\frac{1}{\sqrt{3}}e^{2t}\sin\sqrt{3}t.}$$

8. First we complete the square to get

$$s^2+6s+13 = (s+3)^2+4.$$

Observe that

$$\frac{5s+2}{(s^2+6s+13)^2} = \frac{5(s+3)-13}{((s+3)^2+4)^2}.$$

We now use the frequency shift theorem $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$ to conclude that

$$\mathscr{L}^{-1}\left\{\frac{5(s+3)-13}{((s+3)^2+4)^2}\right\} = e^{-3t}\mathscr{L}^{-1}\left\{\frac{5s-13}{(s^2+4)^2}\right\}.$$

Now let's focus on $\mathscr{L}^{-1}\left\{\dfrac{5s-13}{(s^2+4)^2}\right\}$.

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{5s-13}{(s^2+4)^2}\right\} &= \frac{5}{4}\mathscr{L}^{-1}\left\{\frac{4s}{(s^2+4)^2}\right\} - 13\mathscr{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} \\
&= \frac{5}{4}t\sin 2t - 13\left(\frac{\sin 2t - 2t\cos 2t}{2(2)^3}\right) \\
&= \frac{5}{4}t\sin 2t - \frac{13}{16}\sin 2t + \frac{13}{8}t\cos 2t
\end{aligned}$$

Therefore,

$$\boxed{\mathscr{L}^{-1}\left\{\frac{5s+2}{(s^2+6s+13)^2}\right\} = e^{-3t}\left(\frac{5}{4}\,t\sin 2t - \frac{13}{16}\sin 2t + \frac{13}{8}\,t\cos 2t\right).}$$

9. We can factor the denominator to obtain

$$s^2+4s-5 = (s-1)(s+5).$$

The partial fractions decomposition is of the form

$$\frac{s+3}{(s-1)(s+5)} = \frac{A}{s-1} + \frac{B}{s+5}.$$

Using the cover-up method we obtain A and B as follows.

$$A = \left.\frac{s+3}{\cancel{(s-1)}\,(s+5)}\right|_{s=1} = \frac{2}{3} \quad \text{and} \quad B = \left.\frac{s+3}{(s-1)\,\cancel{(s+5)}}\right|_{s=-5} = \frac{1}{3}$$

Then,

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{s+3}{(s-1)(s+5)}\right\} &= \frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+5}\right\} \\
&= \boxed{\frac{2}{3}\,e^{t} + \frac{1}{3}\,e^{-5t}.}
\end{aligned}$$

10. The partial fractions decomposition is of the form

$$\frac{s+1}{(2s-1)(s+2)} = \frac{A}{2s-1} + \frac{B}{s+2}.$$

Using the cover-up method we obtain A and B as follows.

$$A = \left.\frac{s+1}{\cancel{(2s-1)}\,(s+2)}\right|_{s=1/2} = \frac{3}{5} \quad \text{and} \quad B = \left.\frac{s+1}{(2s-1)\,\cancel{(s+2)}}\right|_{s=-2} = \frac{1}{5}$$

Then,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+1}{(2s-1)(s+2)}\right\} &= \frac{3}{5}\mathscr{L}^{-1}\left\{\frac{1}{2s-1}\right\} + \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}\\ &= \frac{3}{5\cdot 2}\mathscr{L}^{-1}\left\{\frac{1}{s-1/2}\right\} + \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}\\ &= \boxed{\frac{3}{10}\,e^{t/2} + \frac{1}{5}\,e^{-2t}.}\end{aligned}$$

11. We start by factoring the denominator to get $s^4 - s^2 = s^2(s-1)(s+1)$. The partial fractions decomposition is of the form

$$\frac{s^2+s+5}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}.$$

By multiplying both sides by $s^2(s-1)(s+1)$ we get

$$s^2+s+5 = As(s-1)(s+1) + B(s-1)(s+1) + Cs^2(s+1) + Ds^2(s-1).$$

To find the constants, let's assign values to s as follows.

$$\begin{aligned}&s=0 \implies 5 = 0 - B + 0 + 0 \implies B = -5\\ &s=1 \implies 7 = 0 + 0 + 2C + 0 \implies C = 7/2\\ &s=-1 \implies 5 = 0 + 0 + 0 - 2D \implies D = -5/2\\ &s=2 \implies 11 = 6A + 3B + 12C + 4D \implies A = \frac{11 - 3B - -12C - 4D}{6} = -1\end{aligned}$$

Then,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s^2+s+5}{s^4-s^2}\right\} &= -\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 5\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{7}{2}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{5}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= \boxed{-1 - 5t + \frac{7}{2}\,e^{t} - \frac{5}{2}\,e^{-t}.}\end{aligned}$$

12. The partial fractions decomposition is of the form

$$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+c}{s^2+4}.$$

We can use the cover-up method to find A, B, C as follows.

$$A = \left.\frac{1}{\cancel{(s+1)}\,(s^2+4)}\right|_{s=-1} = \frac{1}{5}$$

$$\begin{aligned}(Bs+C)|_{s=2j} &= \left.\frac{1}{(s+1)\cancel{(s^2+4)}}\right|_{s=2j} \\ 2Bj+C &= \frac{1}{1+2j} \\ &= \frac{1}{1+2j}\left(\frac{1-2j}{1-2j}\right) \\ &= \frac{1-2j}{5}.\end{aligned}$$

Since the real and imaginary parts are equal on both sides, we get

$$B=-\frac{1}{5} \quad \text{and} \quad C=\frac{1}{5}.$$

Then,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\} &= \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{5}\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{1}{5\cdot 2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \boxed{\frac{1}{5}e^{-t}-\frac{1}{5}\cos 2t+\frac{1}{10}\sin 2t.}\end{aligned}$$

13. The partial fractions decomposition is of the form

$$\frac{s^2+3s+4}{(s-2)^3}=\frac{A}{s-2}+\frac{B}{(s-2)^2}+\frac{C}{(s-2)^3}.$$

By multiplying both sides by $(s-3)^3$ and expanding, we get

$$\begin{aligned}s^2+3s+4 &= A(s-2)^2+B(s-2)+C \\ &= A(s^2-4s+4)+Bs-2B+C \\ &= As^2+(B-4A)s+(C-2B+4A).\end{aligned}$$

We conclude that

$$A=1, \quad B-4A=3 \implies B=7, \quad C-2B+4A=4 \implies C=14.$$

Then,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s^2+3s+4}{(s-2)^3}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}+7\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}+\frac{14}{2!}\mathscr{L}^{-1}\left\{\frac{2!}{(s-2)^3}\right\} \\ &= \boxed{e^{2t}+7te^{2t}+7t^2e^{2t}.}\end{aligned}$$

14. We factor the denominator to get $s^3+9s=s(s^2+9)$. The partial fractions decomposition is of the form

$$\frac{s^2+s+4}{s(s^2+9)}=\frac{A}{s}+\frac{Bs+C}{s^2+9}.$$

By multiplying both sides by $s(s^2+9)$ and expanding, we get

$$\begin{aligned} s^2+s+4 &= A(s^2+9)+(Bs+C)s \\ &= As^2+9A+Bs^2+Cs \\ &= (A+B)s^2+Cs+9A. \end{aligned}$$

We conclude that

$$9A=4 \implies A=4/9, \quad C=1, \quad A+B=1 \implies B=5/9.$$

Then,

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{s^2+s+4}{s^3+9s}\right\} &= \frac{4}{9}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{5}{9}\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \\ &= \boxed{\frac{4}{9}+\frac{5}{9}\cos 3t+\frac{1}{3}\sin 3t.} \end{aligned}$$

15. The partial fractions decomposition is of the form

$$\frac{3s+2}{s^3(s^2+1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{s^2+1}.$$

By multiplying both sides by $s^3(s^2+1)$ and expanding, we get

$$\begin{aligned} 3s+2 &= As^2(s^2+1)+Bs(s^2+1)+C(s^2+1)+(Ds+E)s^3 \\ &= As^4+As^2+Bs^3+Bs+Cs^2+C+Ds^4+Es^3 \\ 0s^4+0s^3+0s^2+3s+2 &= (A+D)s^4+(B+E)s^3+(A+C)s^2+Bs+C. \end{aligned}$$

We conclude that

$$\begin{aligned} C&=2 \\ B&=3 \\ A+C=0 &\implies A=-2 \\ B+E=0 &\implies E=-3 \\ A+D=0 &\implies D=2. \end{aligned}$$

Then,

$$\boxed{\mathscr{L}^{-1}\left\{\frac{3s+2}{s^3(s^2+1)}\right\}=-2+3t+t^2+2\cos t-3\sin t.}$$

16. First we complete the square to get $s^2+4s+13=(s+2)^2+9$. We can look for a partial fractions decomposition in the form

$$\frac{2s+1}{(s+3)(s^2+4s+13)}=\frac{A}{s+3}+\frac{B(s+2)+C}{(s+2)^2+9}.$$

Method 1. Let's multiply both sides by $(s+3)((s+2)^2+9)$ to get

$$2s+1 = A((s+2)^2+9) + (B(s+2)+C)(s+3).$$

To find the constants A, B, and C, let's assign values to s as follows.

$$\begin{aligned}
s=-3 &\implies -5 = 10A+0 \implies A=-1/2\\
s=-2 &\implies -3 = 9A+C \implies C=-3-9A=3/2\\
s=-1 &\implies -1 = 10A+2B+2C \implies B=(-3-10A-2C)/2=1/2
\end{aligned}$$

Method 2. We can use the cover-up method to find A, B, and C.

$$A = \left.\frac{2s+1}{\cancel{(s+3)}\,(s^2+4s+13)}\right|_{s=-3} = -\frac{1}{2}.$$

Since $s^2+4s+13=(s+2)^2+9=0$ if $s=-2+3j$, then

$$\begin{aligned}
(B(s+2)+C)|_{s=-2+3j} &= \left.\frac{2s+1}{(s+3)\,\cancel{(s^2+4s+13)}}\right|_{s=-2+3j}\\
3Bj+C &= \frac{2(-2+3j)+1}{(-2+3j)+3}\\
&= \frac{-3+6j}{1+3j}\\
&= \left(\frac{-3+6j}{1+3j}\right)\left(\frac{1-3j}{1-3j}\right)\\
3Bj+C &= \frac{15j+15}{10}.
\end{aligned}$$

Since the real and imaginary parts are equal on both sides, we get

$$3B = \frac{15}{10} \implies B=\frac{1}{2} \quad \text{and} \quad C=\frac{15}{10}=\frac{3}{2}.$$

Then,

$$\frac{2s+1}{(s+3)(s^2+4s+13)} = -\frac{1}{2}\left(\frac{1}{s+3}\right)+\frac{1}{2}\left(\frac{s+2}{(s+2)^2+9}\right)+\frac{1}{2}\left(\frac{3}{(s+2)^2+9}\right)$$

and

$$\boxed{\mathscr{L}^{-1}\left\{\frac{2s+1}{(s+3)(s^2+4s+13)}\right\} = -\frac{1}{2}\,e^{3t}+\frac{1}{2}\,e^{-2t}\left(\cos 3t+\sin 3t\right).}$$

2.3 Initial Value Problems

1. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y'(t)\} + 4\mathscr{L}\{y(t)\} = \mathscr{L}\{e^t\}$$

Since $y(0) = 2$, we get

$$(sY(s) - 2) + 4Y(s) = \frac{1}{s-1}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2}{s+4} + \frac{1}{(s+4)(s-1)} = \frac{2s-1}{(s+4)(s-1)}.$$

Using partial fractions, we obtain

$$Y(s) = \frac{9}{5}\left(\frac{1}{s+4}\right) + \frac{1}{5}\left(\frac{1}{s-1}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{9}{5}\, e^{-4t} + \frac{1}{5}\, e^t}$$

2. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y'(t)\} - \mathscr{L}\{y(t)\} = \mathscr{L}\{\sin t\}$$

Since $y(0) = 1$, we get

$$(sY(s) - 1) - Y(s) = \frac{1}{s^2+1}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s-1} + \frac{1}{(s-1)(s^2+1)} = \frac{s^2+2}{(s-1)(s^2+1)}.$$

Using partial fractions, we obtain

$$Y(s) = \frac{3}{2}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{s}{s^2+1}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{3}{2}\, e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t}$$

3. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + 3\mathscr{L}\{y'(t)\} + 2\mathscr{L}\{y(t)\} = \mathscr{L}\{t\} + \mathscr{L}\{1\}$$

Since $y(0) = 1$ and $y'(0) = 0$, we get

$$(s^2Y(s) - s - 0) + 3(sY(s) - 1) + 2Y(s) = \frac{1}{s^2} + \frac{1}{s}.$$

Solving for $Y(s)$ and simplifying gives

$$\begin{aligned} Y(s) &= \frac{s+3}{s^2+3s+2} + \frac{1}{s^2(s^2+3s+2)} + \frac{1}{s(s^2+3s+2)} \\ &= \frac{s^3+3s^2+s+1}{s^2(s+1)(s+2)}. \end{aligned}$$

Using partial fractions, we obtain

$$Y(s) = \frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{1}{4}\left(\frac{1}{s}\right) + 2\left(\frac{1}{s+1}\right) - \frac{3}{4}\left(\frac{1}{s+2}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{1}{2}\,t - \frac{1}{4} + 2e^{-t} - \frac{3}{4}\,e^{-2t}}$$

4. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + 4\mathscr{L}\{y'(t)\} + 13\mathscr{L}\{y(t)\} = 0$$

Since $y(0) = 1$ and $y'(0) = 2$, we get

$$(s^2Y(s) - s - 2) + 4(sY(s) - 1) + 13Y(s) = 0.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s+6}{s^2+4s+13}.$$

By completing the square of the denominator, we get

$$s^2 + 4s + 13 = (s+2)^2 + 9.$$

Therefore,

$$Y(s) = \frac{(s+2)+4}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} + \frac{4}{3}\left(\frac{3}{(s+2)^2+9}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = e^{-2t}\cos 3t + \frac{4}{3}\,e^{-2t}\sin 3t}$$

5. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + 4\mathscr{L}\{y(t)\} = 4\mathscr{L}\{t\} + \mathscr{L}\{8\}$$

Since $y(0) = 4$ and $y'(0) = -1$, we get

$$(s^2Y(s) - 4s + 1) + 4Y(s) = \frac{4}{s^2} + \frac{8}{s}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{4s-1}{s^2+4} + \frac{4}{s^2(s^2+4)} + \frac{8}{s(s^2+4)} = \frac{4s^3 - s^2 + 8s + 4}{s^2(s^2+4)}.$$

Using partial fractions, we obtain

$$Y(s) = \frac{1}{s^2} + 2\left(\frac{1}{s}\right) + 2\left(\frac{s}{s^2+4}\right) - \frac{2}{s^2+4}.$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = t + 2 + \cos 2t - \sin 2t}$$

6. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + \mathscr{L}\{y'(t)\} - 2\mathscr{L}\{y(t)\} = 5\mathscr{L}\{e^{3t}\}$$

Since $y(0) = 1$ and $y'(0) = -4$, we get

$$(s^2Y(s) - s + 4) + (sY(s) - 1) - 2Y(s) = \frac{5}{s-3}.$$

Solving for $Y(s)$ and simplifying gives

$$\begin{aligned} Y(s) &= \frac{s-3}{s^2+s-2} + \frac{5}{(s^2+s-2)(s-3)} \\ &= \frac{(s-3)^2+5}{(s^2+s-2)(s-3)} \\ &= \frac{s^2-6s+14}{(s+2)(s-1)(s-3)}. \end{aligned}$$

Using partial fractions, we obtain

$$Y(s) = 2\left(\frac{1}{s+2}\right) - \frac{3}{2}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = 2e^{-2t} - \frac{3}{2}e^{t} + \frac{1}{2}e^{3t}}$$

7. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + \mathscr{L}\{y'(t)\} - 2\mathscr{L}\{y(t)\} = \mathscr{L}\{e^t\}$$

Since $y(0) = 2$ and $y'(0) = 3$, we get

$$(s^2Y(s) - 2s - 3) + (sY(s) - 2) - 2Y(s) = \frac{1}{s-1}.$$

Solving for $Y(s)$ and simplifying gives

$$\begin{aligned} Y(s) &= \frac{2s+5}{s^2+s-2} + \frac{1}{(s^2+s-2)(s-1)} \\ &= \frac{(2s+5)(s-1)+1}{(s^2+s-2)(s-1)} \\ &= \frac{2s^2+3s-4}{(s+2)(s-1)^2}. \end{aligned}$$

Using partial fractions, we obtain

$$Y(s) = -\frac{2}{9}\left(\frac{1}{s+2}\right) + \frac{20}{9}\left(\frac{1}{s-1}\right) + \frac{1}{3}\left(\frac{1}{(s-1)^2}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = -\frac{2}{9}e^{-2t} + \frac{20}{9}\,e^t + \frac{1}{3}\,te^t}$$

8. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} - 2\mathscr{L}\{y'(t)\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{e^t\}$$

Since $y(0) = 3$ and $y'(0) = 4$, we get

$$(s^2Y(s) - 3s - 4) - 2(sY(s) - 3) + Y(s) = \frac{1}{s-1}.$$

Solving for $Y(s)$ and simplifying gives

$$\begin{aligned} Y(s) &= \frac{3s-2}{s^2-2s+1} + \frac{1}{(s^2-2s+1)(s-1)} \\ &= \frac{3(s-1)+1}{(s-1)^2} + \frac{1}{(s-1)^3} \\ &= 3\left(\frac{1}{s-1}\right) + \frac{1}{(s-1)^2} + \frac{1}{2!}\left(\frac{2!}{(s-1)^3}\right). \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = 3e^t + te^t + \frac{1}{2}\,t^2e^t}$$

9. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + 2\mathscr{L}\{y'(t)\} + 2\mathscr{L}\{y(t)\} = \mathscr{L}\{\cos 2t\}$$

Since $y(0) = 0$ and $y'(0) = 1$, we get

$$(s^2Y(s) - 1) + 2sY(s) + 2Y(s) = \frac{s}{s^2+4}.$$

Solving for $Y(s)$ and simplifying gives

$$\begin{aligned} Y(s) &= \frac{1}{s^2+2s+2} + \frac{s}{(s^2+2s+2)(s^2+4)} \\ &= \frac{s^2+s+4}{(s^2+2s+2)(s^2+4)} \end{aligned}$$

Using partial fractions, we obtain

$$Y(s) = \frac{1}{10}\left(\frac{s+8}{s^2+2s+2}\right) - \frac{1}{10}\left(\frac{s-4}{s^2+4}\right)$$

By completing the square, we get

$$s^2 + 2s + 2 = (s+1)^2 + 1.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{1}{10}\left(\frac{(s+1)+7}{(s+1)^2+1}\right) - \frac{1}{10}\left(\frac{s-4}{s^2+4}\right) \\ &= \frac{1}{10}\left(\frac{s+1}{(s+1)^2+1}\right) + \frac{7}{10}\left(\frac{1}{(s+1)^2+1}\right) - \frac{1}{10}\left(\frac{s}{s^2+4}\right) + \frac{4}{10\cdot 2}\left(\frac{2}{s^2+4}\right) \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{1}{10}e^{-t}\cos t + \frac{7}{10}e^{-t}\sin t - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t}$$

10. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + 4\mathscr{L}\{y(t)\} = \mathscr{L}\{\sin 3t\}$$

Since $y(0) = 2$ and $y'(0) = 1$, we get

$$(s^2Y(s) - 2s - 1) + 4Y(s) = \frac{3}{s^2+9}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2s+1}{s^2+4} + \frac{3}{(s^2+4)(s^2+9)}.$$

Using partial fractions, we obtain

$$\frac{3}{(s^2+4)(s^2+9)} = \frac{3}{5(s^2+4)} - \frac{3}{5(s^2+9)}.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{2s+1}{s^2+4} + \frac{3}{5(s^2+4)} - \frac{3}{5(s^2+9)} \\ &= 2\left(\frac{s}{s^2+4}\right) + \frac{4}{5}\left(\frac{2}{(s^2+4)}\right) - \frac{1}{5}\left(\frac{3}{s^2+9}\right). \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = 2\cos 2t + \frac{4}{5}\sin 2t - \frac{1}{5}\sin 3t}$$

11. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + \omega^2\mathscr{L}\{y(t)\} = \mathscr{L}\{\cos\omega t\}$$

Since $y(0) = 0$ and $y'(0) = 0$, we get

$$s^2Y(s) + \omega^2Y(s) = \frac{s}{s^2+\omega^2}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s}{(s^2+\omega^2)^2} = \frac{1}{2\omega}\left(\frac{2\omega s}{(s^2+\omega^2)^2}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = \frac{1}{2\omega}\,t\sin\omega t}$$

12. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + 2\mathscr{L}\{y'(t)\} + 5\mathscr{L}\{y(t)\} = 3\mathscr{L}\left\{e^{-t}\cos 2t\right\}$$

Since $y(0) = 1$ and $y'(0) = 2$, we get

$$(s^2Y(s) - s - 2) + 2(sY(s) - 1) + 5Y(s) = \frac{3(s+1)}{(s+1)^2+4}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s+4}{s^2+2s+5} + \frac{3(s+1)}{(s^2+2s+5)((s+1)^2+4)}.$$

Since $s^2 + 2s + 5 = (s+1)^2 + 4$, we obtain

$$\begin{aligned} Y(s) &= \frac{(s+1)+3}{(s+1)^2+4} + \frac{3(s+1)}{((s+1)^2+4)^2} \\ &= \frac{s+1}{(s+1)^2+4} + \frac{3}{2}\left(\frac{2}{(s+1)^2+4}\right) + \frac{3(s+1)}{((s+1)^2+4)^2}. \end{aligned}$$

Taking the inverse Laplace transform and using frequency shift property

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\,\mathscr{L}^{-1}\{F(s)\}$$

we get

$$\begin{aligned} y(t) &= e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(2t) + \mathscr{L}^{-1}\left\{\frac{3(s+1)}{((s+1)^2+4)^2}\right\} \\ &= e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(2t) + e^{-t}\,\mathscr{L}^{-1}\left\{\frac{3s}{(s^2+4)^2}\right\} \\ &= e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(2t) + \frac{3}{4}e^{-t}\,\mathscr{L}^{-1}\left\{\frac{4s}{(s^2+4)^2}\right\}. \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = e^{-t}\cos(2t) + \frac{3}{2}\,e^{-t}\sin(2t) + \frac{3}{4}\,te^{-t}\sin(2t)}$$

13. The differential equation of the mass-spring system is

$$x''(t) + 6x'(t) + 5x(t) = F_e(t).$$

(a) We have $F_e(t) = 0$ and the initial conditions are

$$x(0) = 3 \quad \text{and} \quad x'(0) = 1.$$

Take the Laplace transform on both sides of the DE to get

$$(s^2X(s) - 3s - 1) + 6(sX(s) - 3) + 5X(s) = 0.$$

Solving for $X(s)$ gives

$$X(s) = \frac{3s+19}{s^2+6s+5} = \frac{3s+19}{(s+1)(s+5)}.$$

Using partial fractions, we obtain

$$X(s) = \frac{4}{s+1} - \frac{1}{s+5}.$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{x(t) = 4e^{-t} - e^{-5t}}$$

(b) We have $F_e(t) = 30 \sin 2t$ and the initial conditions are

$$x(0) = 0 \quad \text{and} \quad x'(0) = 0.$$

Take the Laplace transform on both sides of the DE to get

$$s^2 X(s) + 6sX(s) + 5X(s) = 30 \left(\frac{2}{s^2+4} \right).$$

Solving for $X(s)$ gives

$$X(s) = \frac{60}{(s^2+6s+5)(s^2+4)} = \frac{60}{(s+1)(s+5)(s^2+4)}.$$

Using partial fractions, we obtain

$$X(s) = 3\left(\frac{1}{s+1}\right) - \frac{15}{29}\left(\frac{1}{s+5}\right) - \frac{72}{29}\left(\frac{s}{s^2+4}\right) + \frac{12}{29 \cdot 2}\left(\frac{2}{s^2+4}\right).$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{x(t) = 3e^{-t} - \frac{15}{29}\, e^{-5t} - \frac{72}{29} \cos 2t + \frac{6}{29} \sin 2t}$$

14. The differential equation of the LR circuit is

$$2i'(t) + 4i(t) = 5e^{-t}$$

with initial condition $i(0) = 0$. Take the Laplace transform on both sides of the DE to get

$$2sI(s) + 4I(s) = \frac{5}{s+1}.$$

Solving for $I(s)$ gives

$$I(s) = \frac{5}{(2s+4)(s+1)}.$$

Using partial fractions, we obtain

$$I(s) = \frac{5}{2(s+1)} - \frac{5}{2(s+2)}.$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{i(t) = \frac{5}{2}\, e^{-t} - \frac{5}{2}\, e^{-2t}}$$

15. The differential equation of the LRC circuit is

$$q''(t) + 2q'(t) + 4q(t) = 50\cos t$$

with initial condition $q'(0) = 0$ and $q(0) = 0$. Take the Laplace transform on both sides of the DE to get

$$s^2Q(s) + 2sQ(s) + 4Q(s) = \frac{50s}{s^2+1}.$$

Solving for $Q(s)$ gives

$$Q(s) = \frac{50s}{(s^2+1)(s^2+2s+4)}.$$

Using partial fractions, we obtain

$$Q(s) = \frac{50}{13}\left(\frac{3s+2}{s^2+1}\right) - \frac{50}{13}\left(\frac{3s+8}{s^2+2s+4}\right).$$

By completing the square, we get

$$s^2 + 2s + 4 = (s+1)^2 + 3.$$

Therefore

$$\begin{aligned} Q(s) &= \frac{50}{13}\left(\frac{3s+2}{s^2+1}\right) - \frac{50}{13}\left(\frac{3(s+1)+5}{(s+1)^2+3}\right) \\ &= \frac{150}{13}\left(\frac{s}{s^2+1}\right) + \frac{100}{13}\left(\frac{1}{s^2+1}\right) - \frac{150}{13}\left(\frac{s+1}{(s+1)^2+3}\right) - \frac{250}{13\sqrt{3}}\left(\frac{\sqrt{3}}{(s+1)^2+3}\right). \end{aligned}$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{q(t) = \frac{150}{13}\cos t + \frac{100}{13}\sin t - \frac{150}{13}e^{-t}\cos\sqrt{3}t - \frac{250}{13\sqrt{3}}e^{-t}\sin\sqrt{3}t}$$

2.4 Step Functions and Impulses

1. We will use the following time shift property.

$$\mathscr{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$$

(a) We have $f(t) = 1 + 2\mathcal{U}(t-1) - 5\mathcal{U}(t-2)$. Then,

$$\boxed{\mathscr{L}\{f(t)\} = \frac{1}{s} + \frac{2e^{-s}}{s} - \frac{5e^{-2s}}{s}.}$$

(b) We have $f(t) = \sin t - \sin t\, \mathcal{U}(t-\pi)$. Then,

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \mathscr{L}\{\sin t\} - \mathscr{L}\{\sin t\, \mathcal{U}(t-\pi)\} \\ &= \frac{1}{s^2+1} - e^{-\pi s}\mathscr{L}\{\sin(t+\pi)\} \\ &= \frac{1}{s^2+1} - e^{-\pi s}\mathscr{L}\{-\sin t\} \\ &= \boxed{\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}.}\end{aligned}$$

(c) We have $f(t) = t - t\mathcal{U}(t-1) + \mathcal{U}(t-1) - \mathcal{U}(t-2)$. Then,

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \mathscr{L}\{t\} - \mathscr{L}\{t\mathcal{U}(t-1)\} + \mathscr{L}\{\mathcal{U}(t-1)\} - \mathscr{L}\{\mathcal{U}(t-2)\} \\ &= \frac{1}{s^2} - e^{-s}\mathscr{L}\{t+1\} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\ &= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\ &= \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}.}\end{aligned}$$

(d) We have $f(t) = t^2\,\mathcal{U}(t-3)$. Then,

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \mathscr{L}\left\{t^2\,\mathcal{U}(t-3)\right\} \\ &= e^{-3s}\mathscr{L}\left\{(t+3)^2\right\} \\ &= e^{-3s}\mathscr{L}\left\{t^2+6t+9\right\} \\ &= \boxed{e^{-s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).}\end{aligned}$$

2. We will use the following time shift property.

$$\mathscr{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)\,\mathcal{U}(t-a)$$

(a) If $F(s) = \dfrac{1}{s+3}$, then $f(t) = e^{-3t}$. Therefore,

$$\boxed{\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s+3}\right\} = e^{-3(t-2)}\,\mathcal{U}(t-2).}$$

(b) If $F(s) = \dfrac{1}{s^2+4} = \dfrac{1}{2}\left(\dfrac{2}{s^2+4}\right)$, then $f(t) = \frac{1}{2}\sin 2t$. Therefore,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\} &= \frac{1}{2}\sin 2(t-2\pi)\,\mathcal{U}(t-2\pi) \\ &= \boxed{\frac{1}{2}\sin 2t\,\mathcal{U}(t-2\pi).}\end{aligned}$$

(c) Let $F(s) = \dfrac{1}{(s-1)(s+2)}$. Using partial fractions, we get

$$F(s) = \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right).$$

Since $f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{3}(e^t - e^{-2t})$, then

$$\boxed{\mathscr{L}^{-1}\left\{\frac{e^{-s}}{(s-1)(s+2)}\right\} = \frac{1}{3}\left(e^{(t-1)} - e^{-2(t-1)}\right)\mathcal{U}(t-1).}$$

(d) Let $F(s) = \dfrac{1}{s(s^2+1)}$. Using partial fractions, we get

$$F(s) = \frac{1}{s} - \frac{s}{s^2+1}.$$

Since $f(t) = \mathscr{L}^{-1}\{F(s)\} = 1 - \cos t$, then

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+1)}\right\} &= (1 - \cos(t-\pi))\,\mathcal{U}(t-\pi)\\ &= \boxed{(1+\cos t)\,\mathcal{U}(t-\pi).}\end{aligned}$$

3. The IVP is

$$y'(t) + 2y(t) = 1 - \mathcal{U}(t-1), \quad y(0) = 3.$$

Take the Laplace transform on both sides of the DE to get

$$(sY(s) - 3) + 2Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{3}{s+2} + \frac{1}{s(s+2)}\left(1 - e^{-s}\right).$$

Let $G(s) = \dfrac{1}{s(s+2)}$. Using partial fractions we obtain

$$G(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right) \implies g(t) = \frac{1}{2}\left(1 - e^{-2t}\right).$$

Since

$$Y(s) = \frac{3}{s+2} + G(s) - e^{-s}G(s),$$

we conclude that

$$\begin{aligned} y(t) &= 3e^{-2t} + g(t) - g(t-1)\,\mathcal{U}(t-1) \\ &= 3e^{-2t} + \frac{1}{2}\left(1 - e^{-2t}\right) - \frac{1}{2}\left(1 - e^{-2(t-1)}\right)\mathcal{U}(t-1) \\ &= \boxed{\frac{1}{2} + \frac{5}{2}\,e^{-2t} - \frac{1}{2}\left(1 - e^{-2(t-1)}\right)\mathcal{U}(t-1).} \end{aligned}$$

We can also express $y(t)$ as a piecewise defined function as follows.

$$y(t) = \begin{cases} \frac{1}{2} + \frac{5}{2}\,e^{-2t}, & 0 \le t < 1 \\ \frac{5}{2}\,e^{-2t} + \frac{1}{2}\,e^{-2(t-1)}, & t \ge 1 \end{cases}$$

4. The IVP is

$$y''(t) + y(t) = \sin t\,\mathcal{U}(t-\pi), \quad y(0) = 1,\ y'(0) = 0.$$

Take the Laplace transform on both sides of the DE to get

$$\begin{aligned} (s^2 Y(s) - s) + Y(s) &= \mathscr{L}\{\sin t\,\mathcal{U}(t-\pi)\} \\ &= e^{-\pi s}\mathscr{L}\{\sin(t+\pi)\} \\ &= e^{-\pi s}\mathscr{L}\{-\sin t\} \\ &= -\frac{e^{-\pi s}}{s^2+1}. \end{aligned}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{s}{s^2+1} - \frac{e^{-\pi s}}{(s^2+1)^2}.$$

Since

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{\sin t - t\cos t}{2},$$

we get

$$\begin{aligned} y(t) &= \cos t - \left(\frac{\sin(t-\pi) - (t-\pi)\cos(t-\pi)}{2}\right)\mathcal{U}(t-\pi) \\ &= \boxed{\cos t + \left(\frac{\sin t - (t-\pi)\cos t}{2}\right)\mathcal{U}(t-\pi).} \end{aligned}$$

We can also express $y(t)$ as a piecewise defined function as follows.

$$y(t) = \begin{cases} \cos t, & 0 \le t < \pi \\ (1 - \frac{1}{2}(t-\pi))\cos t + \frac{1}{2}\sin t, & t \ge \pi \end{cases}$$

5. The IVP is

$$y''(t) + y'(t) - 2y(t) = 1 - 2\mathcal{U}(t-3), \quad y(0) = 0,\ y'(0) = 0.$$

Take the Laplace transform on both sides of the DE to get

$$s^2Y(s) + sY(s) - 2Y(s) = \frac{1}{s} - \frac{2e^{-3s}}{s}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1-2e^{-3s}}{s(s^2+s-2)} = \frac{1-2e^{-3s}}{s(s-1)(s+2)}.$$

Let $G(s) = \dfrac{1}{s(s-1)(s+2)}$. Using partial fractions we obtain

$$G(s) = \frac{1}{3(s-1)} + \frac{1}{6(s+2)} - \frac{1}{2s} \implies g(t) = \frac{1}{3}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2}.$$

Since

$$Y(s) = G(s) - 2e^{-3s}G(s),$$

we conclude that

$$\boxed{y(t) = \frac{1}{3}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2} - \left(\frac{2}{3}e^{(t-3)} + \frac{1}{3}e^{-2(t-3)} - 1\right)\mathcal{U}(t-3).}$$

We can also express $y(t)$ as a piecewise defined function as follows.

$$y(t) = \begin{cases} \frac{1}{3}e^t + \frac{1}{6}e^{-2t} - \frac{1}{2}, & 0 \le t < 3 \\ \frac{1}{3}\left(e^t - 2e^{(t-3)}\right) + \frac{1}{6}\left(e^{-2t} - 2e^{-2(t-3)}\right) + \frac{1}{2}, & t \ge 3 \end{cases}$$

6. The IVP is

$$y''(t) + 4y(t) = 1 + \mathcal{U}(t-\pi) - 2\mathcal{U}(t-2\pi), \quad y(0) = 0,\ y'(0) = 2.$$

Take the Laplace transform on both sides of the DE to get

$$(s^2Y(s) - 2) + 4Y(s) = \frac{1}{s} + \frac{e^{-\pi s}}{s} - \frac{2e^{-2\pi s}}{s}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2}{s^2+4} + \frac{1 + e^{-\pi s} - 2e^{-2\pi s}}{s(s^2+4)}.$$

Let $G(s) = \dfrac{1}{s(s^2+4)}$. Using partial fractions we obtain

$$G(s) = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \implies g(t) = \frac{1}{4}\left(1 - \cos 2t\right).$$

Since
$$Y(s) = \frac{2}{s^2+4} + G(s) + e^{-\pi s}G(s) - 2e^{-2\pi s}G(s)$$
we conclude that
$$y(t) = \sin 2t + \frac{1}{4}\left(1-\cos 2t\right) + \frac{1}{4}\left(1-\cos 2(t-\pi)\right)\mathcal{U}(t-\pi) - \frac{1}{2}\left(1-\cos 2(t-2\pi)\right)\mathcal{U}(t-2\pi).$$
Simplifying gives the answer.
$$\boxed{y(t) = \sin 2t + \frac{1}{4}\left(1-\cos 2t\right) + \frac{1}{4}\left(1-\cos 2t\right)\mathcal{U}(t-\pi) - \frac{1}{2}\left(1-\cos 2t\right)\mathcal{U}(t-2\pi)}$$
We can also express $y(t)$ as a piecewise defined function as follows.
$$y(t) = \begin{cases} \sin 2t + \frac{1}{4}\left(1-\cos 2t\right), & 0 \le t < \pi \\ \sin 2t + \frac{1}{2}\left(1-\cos 2t\right), & \pi \le t < 2\pi \\ \sin 2t, & t \ge 2\pi \end{cases}$$

7. Take the Laplace transform on both sides of the differential equation.
$$\mathscr{L}\{y'(t)\} + 3\mathscr{L}\{y(t)\} = \mathscr{L}\{\delta(t-1)\}$$
Since $y(0) = 2$, we get
$$(sY(s) - 2) + 3Y(s) = e^{-s}.$$
Solving for $Y(s)$ gives
$$Y(s) = \frac{2}{s+3} + \frac{e^{-s}}{s+3}.$$
Taking the inverse Laplace transform gives the answer.
$$\boxed{y(t) = 2e^{-3t} + e^{-3(t-1)}\mathcal{U}(t-1)}$$

8. Take the Laplace transform on both sides of the differential equation.
$$\mathscr{L}\{y''(t)\} + 4\mathscr{L}\{y'(t)\} + 13\mathscr{L}\{y(t)\} = \mathscr{L}\{\delta(t-\pi)\}$$
Since $y(0) = 2$ and $y'(0) = 1$, we get
$$(s^2Y(s) - 2s - 1) + 4(sY(s) - 2) + 13Y(s) = e^{-\pi s}.$$
Solving for $Y(s)$ gives
$$Y(s) = \frac{2s+9}{s^2+4s+13} + \frac{e^{-\pi s}}{s^2+4s+13}.$$

By completing the square, we get

$$s^2 + 4s + 13 = (s+2)^2 + 9.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{2s+9}{(s+2)^2+9} + \frac{e^{-\pi s}}{(s+2)^2+9} \\ &= \frac{2(s+2)+5}{(s+2)^2+9} + \frac{e^{-\pi s}}{(s+2)^2+9} \\ &= 2\left(\frac{s+2}{(s+2)^2+9}\right) + \frac{5}{3}\left(\frac{3}{(s+2)^2+9}\right) + \frac{e^{-\pi s}}{3}\left(\frac{3}{(s+2)^2+9}\right). \end{aligned}$$

Taking the inverse Laplace transform gives

$$y(t) = 2e^{-2t}\cos 3t + \frac{5}{3}e^{-2t}\sin 3t + \frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)\,\mathcal{U}(t-\pi).$$

Since $\sin 3(t-\pi) = -\sin 3t$, we obtain the answer

$$\boxed{y(t) = 2e^{-2t}\cos 3t + \frac{5}{3}e^{-2t}\sin 3t - \frac{1}{3}e^{-2(t-\pi)}\sin 3t\,\mathcal{U}(t-\pi).}$$

9. Take the Laplace transform on both sides of the differential equation.

$$\mathscr{L}\{y''(t)\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{\delta(t - \tfrac{\pi}{2})\} + 2\,\mathscr{L}\{\delta(t-\pi)\}$$

Since $y(0) = 3$ and $y'(0) = -1$, we get

$$(s^2 Y(s) - 3s + 1) + Y(s) = e^{-\frac{\pi}{2}s} + 2e^{-\pi s}.$$

Solving for $Y(s)$ gives

$$\begin{aligned} Y(s) &= \frac{3s-1}{s^2+1} + \frac{e^{-\frac{\pi}{2}s} + 2e^{-\pi s}}{s^2+1} \\ &= 3\left(\frac{s}{s^2+1}\right) - \frac{1}{s^2+1} + e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1}\right) + 2e^{-\pi s}\left(\frac{1}{s^2+1}\right). \end{aligned}$$

Taking the inverse Laplace transform gives

$$y(t) = 3\cos t - \sin t + \sin(t - \tfrac{\pi}{2})\,\mathcal{U}(t - \tfrac{\pi}{2}) + 2\sin(t-\pi)\,\mathcal{U}(t-\pi).$$

Since

$$\sin(t - \tfrac{\pi}{2}) = -\cos t \quad \text{and} \quad \sin(t-\pi) = -\sin t,$$

we obtain the answer

$$\boxed{y(t) = 3\cos t - \sin t - \cos t\,\mathcal{U}(t - \tfrac{\pi}{2}) - 2\sin t\,\mathcal{U}(t-\pi).}$$

10. We will use the formula

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

Since

$$f(t) = \begin{cases} 1, & 0 \le t < a \\ 0, & a \le t < 2a \end{cases}$$

and $T = 2a$, we have

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt \\ &= \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-st} f(t)\, dt + \int_a^{2a} e^{-st} f(t)\, dt \right) \\ &= \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-st} (1)\, dt + \int_a^{2a} e^{-st} (0)\, dt \right) \\ &= \frac{1}{1-e^{-2as}} \int_0^a e^{-st}\, dt \\ &= \left(\frac{1}{1-e^{-2as}} \right) \left(\frac{1-e^{-as}}{s} \right) \\ &= \left(\frac{1}{(1+e^{-as})\cancel{(1-e^{-as})}} \right) \left(\frac{\cancel{1-e^{-as}}}{s} \right) \\ &= \boxed{\frac{1}{s(1+e^{-as})}}. \end{aligned}$$

11. Since $V(t) = 2 - 2\mathcal{U}(t-1)$, the differential equation of the LR circuit is

$$i'(t) + 4i(t) = 2 - 2\mathcal{U}(t-1)$$

with initial conditions $i(0) = 0$. Take the Laplace transform on both sides of the DE to get

$$sI(s) + 4I(s) = \frac{2}{s} - \frac{2e^{-s}}{s}.$$

Solving for $I(s)$ gives

$$I(s) = \frac{2}{s(s+4)} - \frac{2e^{-s}}{s(s+4)}.$$

Let $G(s) = \dfrac{2}{s(s+4)}$. Using partial fractions, we obtain

$$G(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+4} \right) \implies g(t) = \frac{1}{2}\left(1 - e^{-4t}\right).$$

Since

$$I(s) = G(s) - e^{-s} G(s),$$

we can take the inverse Laplace transform to get the current.

(a)

$$\boxed{i(t) = \frac{1}{2}\left(1 - e^{-4t}\right) - \frac{1}{2}\left(1 - e^{-4(t-1)}\right)\mathcal{U}(t-1)}$$

(b) We can express $i(t)$ as follows.

$$i(t) = \begin{cases} \frac{1}{2}\left(1 - e^{-4t}\right), & 0 \leq t < 1 \\ \frac{1}{2}\left(e^{-4(t-1)} - e^{-4t}\right), & t \geq 1 \end{cases}$$

For $t = 1.5$, we get

$$i(1.5) = \frac{1}{2}\left(e^{-4(1.5-1)} - e^{-4(1.5)}\right) \approx 0.0664.$$

At $t = 1.5$ seconds, the current is about $0.0664\,\text{A}$.

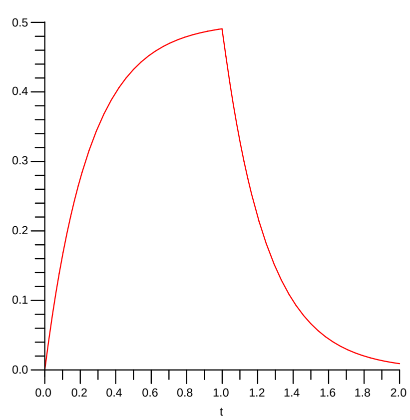
(c) For $t \geq 1$, we have

$$i(t) = \frac{1}{2}\left(e^{-4(t-1)} - e^{-4t}\right).$$

Therefore,

$$\boxed{\lim_{t\to\infty} i(t) = 0.}$$

(d) The graph of $i(t)$ is the following.



Observe that the current $i(t)$ is continuous even though the voltage is discontinuous at $t = 1$.

12. We have

$$\delta_\varepsilon(t-a) = \frac{1}{\varepsilon}\mathcal{U}(t-a) - \frac{1}{\varepsilon}\mathcal{U}(t-(a+\varepsilon)).$$

(a) Taking Laplace transform, we get

$$\begin{aligned}\mathscr{L}\{\delta_\varepsilon(t-a)\} &= \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s}\right) - \frac{1}{\varepsilon}\left(\frac{e^{-(a+\varepsilon)s}}{s}\right) \\ &= \frac{e^{-as}}{s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon}\right).\end{aligned}$$

(b) We will use l'Hôpital's rule to compute the limit.

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathscr{L}\{\delta_\varepsilon(t-a)\} &= \lim_{\varepsilon\to 0}\frac{e^{-as}}{s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon}\right) \\ &= \frac{e^{-as}}{s}\left(\lim_{\varepsilon\to 0}\frac{1-e^{-\varepsilon s}}{\varepsilon}\right) \\ &= \frac{e^{-as}}{s}\left(\lim_{\varepsilon\to 0}\frac{se^{-\varepsilon s}}{1}\right) \\ &= \frac{e^{-as}}{\not{s}}(\not{s}) \\ &= e^{-as}\end{aligned}$$

2.5 Convolution

1. We will use the convolution theorem.

$$\mathscr{L}\{f(t)*g(t)\} = F(s)G(s)$$

(a) $\mathscr{L}\{1*t^4\} = \mathscr{L}\{1\}\,\mathscr{L}\{t^4\} = \frac{1}{s}\left(\frac{4!}{s^5}\right) = \boxed{\frac{24}{s^6}}$

(b) Observe that $\int_0^t \cos\theta\, d\theta = 1*\cos t$.

$$\begin{aligned}\mathscr{L}\left\{\int_0^t \cos\theta\, d\theta\right\} &= \mathscr{L}\{1*\cos t\} \\ &= \mathscr{L}\{1\}\,\mathscr{L}\{\cos t\} \\ &= \frac{1}{\not{s}}\left(\frac{\not{s}}{s^2+1}\right) \\ &= \boxed{\frac{1}{s^2+1}}\end{aligned}$$

(c) $\mathscr{L}\{e^t*\sin t\} = \mathscr{L}\{e^t\}\,\mathscr{L}\{\sin t\} = \left(\frac{1}{s-1}\right)\left(\frac{1}{s^2+1}\right) = \boxed{\frac{1}{(s-1)(s^2+1)}}$

(d) Observe that $\int_0^t \cos\theta \sin(t-\theta)\, d\theta = \cos t * \sin t$.

$$\begin{aligned}\mathscr{L}\left\{\int_0^t \cos\theta\sin(t-\theta)\,d\theta\right\} &= \mathscr{L}\{\cos t * \sin t\}\\ &= \left(\frac{s}{s^2+1}\right)\left(\frac{1}{s^2+1}\right)\\ &= \boxed{\frac{s}{(s^2+1)^2}}\end{aligned}$$

2. We will use the convolution theorem.

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(\theta)g(t-\theta)\,d\theta$$

(a) Let $F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{s-1}$, then

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = 1 \quad \text{and} \quad g(t) = \mathscr{L}^{-1}\{G(s)\} = e^t.$$

Therefore,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} &= \mathscr{L}^{-1}\{F(s)G(s)\}\\ &= f(t) * g(t)\\ &= 1 * e^t\\ &= \int_0^t e^\theta\,d\theta\\ &= e^\theta\Big|_0^t\\ &= \boxed{e^t - 1.}\end{aligned}$$

(b) Let $F(s) = \frac{1}{s+1}$ and $G(s) = \frac{1}{s-3}$, then

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = e^{-t} \quad \text{and} \quad g(t) = \mathscr{L}^{-1}\{G(s)\} = e^{3t}.$$

Therefore,

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{(s+1)(s-3)}\right\} &= \mathscr{L}^{-1}\{F(s)G(s)\} \\
&= f(t) * g(t) \\
&= e^{-t} * e^{3t} \\
&= \int_0^t e^{-(t-\theta)} e^{3\theta}\, d\theta \\
&= e^{-t}\int_0^t e^{4\theta}\, d\theta \\
&= e^{-t}\left(\frac{e^{4\theta}}{4}\right)\Bigg|_0^t \\
&= e^{-t}\left(\frac{e^{4t}-1}{4}\right) \\
&= \boxed{\frac{e^{3t}-e^{-t}}{4}.}
\end{aligned}$$

(c) Let $F(s) = \dfrac{1}{s}$ and $G(s) = \dfrac{1}{s^2+1}$, then

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = 1 \quad \text{and} \quad g(t) = \mathscr{L}^{-1}\{G(s)\} = \sin t.$$

Therefore,

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \mathscr{L}^{-1}\{F(s)G(s)\} \\
&= f(t) * g(t) \\
&= 1 * \sin t \\
&= \int_0^t \sin\theta\, d\theta \\
&= -\cos\theta\Big|_0^t \\
&= \boxed{1-\cos t.}
\end{aligned}$$

(d) Let $F(s) = \dfrac{1}{s}$ and $G(s) = \dfrac{1}{s(s^2+1)}$, then

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = 1 \quad \text{and} \quad g(t) = \mathscr{L}^{-1}\{G(s)\} = 1-\cos t.$$

Therefore,

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \mathscr{L}^{-1}\{F(s)G(s)\} \\
&= f(t) * g(t) \\
&= 1 * (1-\cos t \\
&= \int_0^t 1 - \cos\theta \, d\theta \\
&= (\theta - \sin\theta)\Big|_0^t \\
&= \boxed{t - \sin t.}
\end{aligned}$$

3. Since $\mathscr{L}\{\sin t\} = \dfrac{1}{s^2+1}$, then

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \sin t * \sin t = \int_0^t \sin\theta \sin(t-\theta)\, d\theta$$

By using the identity

$$\sin\alpha \sin\beta = \frac{1}{2}\big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\big)$$

with $\alpha = \theta$ and $\beta = t - \theta$, we get

$$\begin{aligned}
\sin\theta\sin(t-\theta) &= \frac{1}{2}\big(\cos(\theta-(t-\theta)) - \cos(\theta+(t-\theta))\big) \\
&= \frac{1}{2}\big(\cos(2\theta - t) - \cos t\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \frac{1}{2}\int_0^t (\cos(2\theta-t) - \cos t)\, d\theta \\
&= \frac{1}{2}\left(\frac{\sin(2\theta-t)}{2} - \theta\cos t\right)\Bigg|_0^t \\
&= \frac{1}{2}\left(\frac{\sin t}{2} - t\cos t\right) - \frac{1}{2}\left(\frac{\sin(-t)}{2} - 0\right) \\
&= \frac{1}{2}(\sin t - t\cos t)
\end{aligned}$$

We can now apply the scaling property

$$\mathscr{L}\{f(\omega t)\} = \frac{1}{\omega}F\left(\frac{s}{\omega}\right)$$

to the function $f(t) = \frac{1}{2}(\sin t - t\cos t)$ to get

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{2}(\sin \omega t - \omega t \cos \omega t)\right\} &= \frac{1}{\omega}\left(\frac{1}{\left(\left(\frac{s}{\omega}\right)^2 + 1\right)^2}\right) \\ &= \frac{\omega^3}{(s^2+\omega^2)^2}.\end{aligned}$$

We can conclude that

$$\boxed{\mathscr{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} = \frac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}.}$$

4. First observe that

$$\int_0^t e^{-\theta} f(t-\theta)\, d\theta = e^{-t} * f(t).$$

The equation is then equivalent to

$$f(t) = t + e^{2t} + e^{-t} * f(t).$$

Take Laplace transform on both sides to get

$$F(s) = \frac{1}{s^2} + \frac{1}{s-2} + \frac{F(s)}{s+1}.$$

Solve for $F(s)$ to get

$$F(s) = \frac{1}{s^2} + \frac{1}{s^3} + \frac{s+1}{s(s-2)}$$

Using partial fractions, we obtain

$$\frac{s+1}{s(s-2)} = \frac{-1/2}{s} + \frac{3/2}{s-2}.$$

Therefore,

$$F(s) = \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{2}\left(\frac{1}{s}\right) + \frac{3}{2}\left(\frac{1}{s-2}\right)$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{f(t) = t + \frac{t^2}{2} - \frac{1}{2} + \frac{3}{2}e^{2t}}$$

5. The equation is of the form

$$f(t) * f(t) = t^3.$$

Take Laplace transform to get

$$F(s) \cdot F(s) = F(s)^2 = \frac{6}{s^4} \implies F(s) = \pm\frac{\sqrt{6}}{s^2}.$$

By taking the inverse Laplace transform we obtain the two answers

$$\boxed{f(t) = \pm\sqrt{6}t.}$$

6. First observe that

$$\int_0^t e^{-2\theta} y(t-\theta)\, d\theta = e^{-2t} * y(t).$$

The equation is then equivalent to

$$y'(t) + e^{-2t} * y(t) = 1.$$

Since $y(0) = 0$, we can take Laplace transform on both sides to get

$$sY(s) + \frac{Y(s)}{s+2} = \frac{1}{s}.$$

Solve for $Y(s)$ to get

$$Y(s) = \frac{s+2}{s(s+1)^2}.$$

Using partial fractions, we deduce

$$Y(s) = \frac{2}{s} - \frac{2}{s+1} - \frac{1}{(s+1)^2}.$$

Taking the inverse Laplace transform gives the answer.

$$\boxed{y(t) = 2 - 2e^{-t} - te^{-t}}$$

Appendix A

Formulas and Properties

A.1 Table of Laplace Transforms

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$	$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1	$\dfrac{1}{s}$	e^{at}	$\dfrac{1}{s-a}$
t^n	$\dfrac{n!}{s^{n+1}}$	$e^{at}t^n$	$\dfrac{n!}{(s-a)^{n+1}}$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$	$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$t\sin\omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$	$t\cos\omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\dfrac{\sin\omega t-\omega t\cos\omega t}{2\omega^3}$	$\dfrac{1}{(s^2+\omega^2)^2}$	$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$
$\delta(t)$	1	$\delta(t-a)$	e^{-as}

A.2 Properties of Laplace Transforms

1. **Definition**

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

2. **Linearity**

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\}$$

3. **Time differentiation**

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0) \qquad \mathscr{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$$

4. **Frequency differentiation**

$$\mathscr{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

5. **Time shift**

$$\mathscr{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as} \mathscr{L}\{f(t+a)\}$$
$$\mathscr{L}^{-1}\left\{e^{-as} F(s)\right\} = f(t-a)\mathcal{U}(t-a)$$

6. **Frequency shift**

$$\mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

7. **Convolution**

$$f(t) * g(t) = \int_0^t f(\theta) g(t-\theta)\, d\theta \implies \mathscr{L}\{f(t) * g(t)\} = F(s)G(s)$$

8. **Integration**

$$\mathscr{L}\left\{\int_0^t f(\theta)\, d\theta\right\} = \frac{F(s)}{s}$$

9. **Periodic function**

$$f(t) \text{ has period } T \implies \mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

10. **Scaling**

$$\mathscr{L}\{f(\omega t)\} = \frac{1}{\omega} F\left(\frac{s}{\omega}\right)$$

A.3 Trigonometric Identities

1. Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

2. Symmetry Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

3. Sum and Difference Identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

4. Double-Angle Identities

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1\end{aligned} \qquad \begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ \tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

5. Power-Reducing Identities

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

6. Product-To-Sum Identities

$$\begin{aligned}\sin\alpha\cos\beta &= \frac{1}{2}\big(\sin(\alpha-\beta) + \sin(\alpha+\beta)\big) \\ \sin\alpha\sin\beta &= \frac{1}{2}\big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\big) \\ \cos\alpha\cos\beta &= \frac{1}{2}\big(\cos(\alpha-\beta) + \cos(\alpha+\beta)\big)\end{aligned}$$

Appendix B

Partial Fractions

B.1 Partial Fractions

Consider a function $F(s)$ expressed as a quotient of two polynomials

$$F(s) = \frac{P(s)}{Q(s)}$$

where $Q(s)$ has a degree smaller than the degree of $P(s)$. The method of partial fractions can be summarized as follows.

1. Completely factor $Q(s)$ into factors of the form

$$(ps+q)^m \quad \text{and} \quad (as^2+bs+c)^n$$

   where as^2+bs+c is an irreducible quadratic.

2. For *each* factor of the form $(ps+q)^m$, the partial fractions decomposition must include the following m terms.

$$\frac{A_1}{ps+q} + \frac{A_2}{(ps+q)^2} + \cdots + \frac{A_m}{(ps+q)^m}.$$

3. For *each* factor of the form $(as^2+bs+c)^n$, the partial fractions decomposition must include the following n terms.

$$\frac{B_1 s + C_1}{as^2+bs+c} + \frac{B_2 s + C_2}{(as^2+bs+c)^2} + \cdots + \frac{B_n s + C_n}{(as^2+bs+c)^n}.$$

4. Find the values of all the constants.

B.2 Cover-up Method

There are several different ways to determine the constants in a partial fractions decomposition. When the denominator has distinct roots, we can use the cover-up method. Let's illustrate it with some examples.

Example 1. Consider

$$\frac{s^2+5s+4}{(s-1)(s-3)(s+2)} = \frac{A}{s-1} + \frac{B}{s-3} + \frac{C}{s+2} \tag{B.1}$$

To find A, we could use the following two steps.

1. Multiply both sides of equation (B.1) by $(s-1)(s-3)(s+1)$ to obtain

$$s^2+5s+4 = A(s-3)(s+2) + B(s-1)(s+1) + C(s-1)(s-3).$$

2. Set $s=1$ and solve for A.

$$1+5+4 = A(-2)(3)+0+0 \implies A = -\frac{5}{3}$$

The cover-up method combines these two steps in a single one as follows. To find A, we cover up $(s-1)$ and set $s=1$ in the left-hand side of (B.1).

$$A = \left.\frac{s^2+5s+4}{\cancel{(s-1)}(s-3)(s+2)}\right|_{s=1} = -\frac{5}{3}$$

We can find constants B and C in the same way. To find B, we cover up $(s-3)$ and set $s=3$.

$$B = \left.\frac{s^2+5s+4}{(s-1)\cancel{(s-3)}(s+2)}\right|_{s=3} = \frac{14}{5}$$

To find C, we cover up $(s+2)$ and set $s=-2$.

$$C = \left.\frac{s^2+5s+4}{(s-1)(s-3)\cancel{(s+2)}}\right|_{s=-2} = -\frac{2}{15}$$

If the denominator includes irreducible quadratics terms, the cover-up method works if we use complex numbers as in the following examples.

Example 2. Consider

$$\frac{s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

Since $s+1=0$ if $s=-1$, then

$$A = \left.\frac{s}{\cancel{(s+1)}\,(s^2+4)}\right|_{s=-1} = -\frac{1}{5}$$

Since $s^2+4=0$ if $s=2j$, then

$$\begin{aligned}(Bs+C)|_{s=2j} &= \left.\frac{s}{(s+1)\,\cancel{(s^2+4)}}\right|_{s=2j}\\ 2Bj+C &= \frac{2j}{1+2j}\\ &= \frac{2j}{(1+2j)}\left(\frac{1-2j}{1-2j}\right)\\ &= \frac{2j+4}{5}.\end{aligned}$$

The real and imaginary parts of the complex numbers on both sides are equal. We conclude that

$$2B = \frac{2}{5} \implies B = \frac{1}{5}$$

and

$$C = \frac{4}{5}.$$

Example 3. Consider

$$F(s) = \frac{s^2}{(s+2)(s^2+6s+13)}.$$

By completing the square, we get

$$s^2 + 6s + 13 = (s+3)^2 + 4.$$

We can look for a partial fractions decomposition of $F(s)$ in the form

$$\frac{s^2}{(s+2)(s^2+6s+13)} = \frac{A}{s+2} + \frac{B(s+3)+C}{(s+3)^2+4}.$$

Observe that by using $B(s+3)+C$ instead $Bs+C$, we make it a little bit easier since we do not have to solve a linear system to find B and C.

Since $s+2=0$ if $s=-2$, then

$$A = \left.\frac{s^2}{\cancel{(s+2)}\,(s^2+6s+13)}\right|_{s=-2} = \frac{4}{5}$$

Since $s^2+6s+13 = (s+3)^2+4 = 0$ if $s = -3+2j$, then

$$\begin{aligned}
(B(s+3)+C)|_{s=-3+2j} &= \left.\frac{s^2}{(s+2)\,\cancel{(s^2+6s+13)}}\right|_{s=-3+2j} \\
2Bj + C &= \frac{(-3+2j)^2}{(-3+2j)+2} \\
&= \frac{5-12j}{-1+2j} \\
&= \frac{5-12j}{-1+2j}\left(\frac{-1-2j}{-1-2j}\right) \\
2Bj + C &= \frac{2j-29}{5}.
\end{aligned}$$

The real and imaginary parts of the complex numbers on both sides are equal. We conclude that

$$2B = \frac{2}{5} \implies B = \frac{1}{5}$$

and

$$C = -\frac{29}{5}.$$

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