CAMOSUN
COLLEGE

Introduction to Calculus

Contents

Chapter 1

Introduction to Calculus

1.1 Introduction

1.1.1 Origin of Calculus

The development of *Calculus* by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibnitz (1646–1716) is one of the most important achievements in the history of science and mathematics.

Newton is without doubt one of the greatest mathematicians of all time. In his efforts to find a mathematical method that could explain universal gravitation, he devised what he called the *method of fluxions*. Today we call it *differential and integral calculus*.

Newton was a private and secretive man, and for the most part kept his discoveries for himself. He did not publish much, and the majority of his works, like his famous *Philosophiae Naturalis Principia Mathematica*, had to be dragged out of him by the persistence of his friends.

It is now well established that Newton and Leibnitz developed their own form of calculus independently, that Newton was first by about 10 years but did not publish, and that Leibnitz's papers of 1684 and 1686 were the earliest publications on the subject.

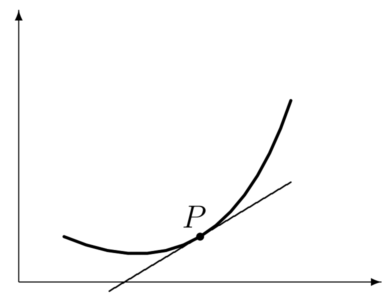
If you are interested in finding out more about Newton and Leibnitz, or the history of mathematics in general, consult the following website:

`http://www-history.mcs.st-and.ac.uk/history`

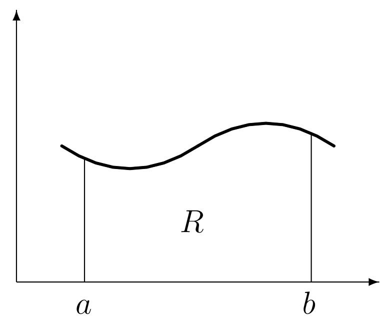
1.1.2 The Two Branches of Calculus

There are two basic geometric problems that call for the use of calculus:

- Finding the slope of the tangent line to a curve at a given point.
- Finding the area between a curve and the x-axis for $a \leq x \leq b$.



What is the slope of the tangent at P?



What is the area of the region R?

We will examine the close relationship between the slope problem and the problem of determining the rate at which a variable is changing as compared to another variable. The portion of calculus concerned with this problem is called *differential calculus*. It relies on the concept of the *derivative* of a function. You will eventually see that the derivative of a function is defined in terms of a more fundamental concept – the concept of a *limit*.

The area problem is related to the problem of finding a variable quantity whose rate of change is known. The part of calculus concerned with these ideas is called *integral calculus* and will not be covered here. It is studied in first year calculus.

1.2 Secant and Tangent Lines

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the graph of $y = f(x)$. The line joining these two points is called a **secant line** and has a slope given by

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}.$$

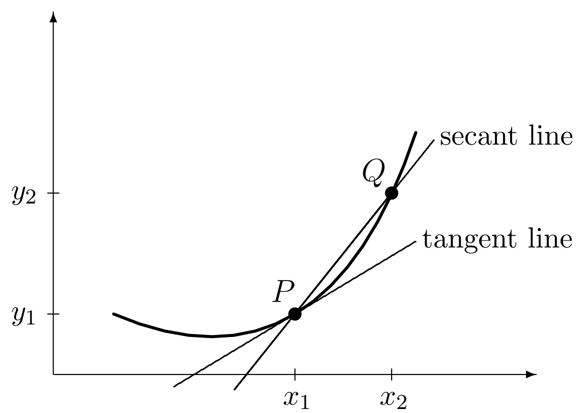


Figure 1.1: **Secant line** joining P and Q and **tangent line** at point P.

If we let $h = x_2 - x_1$, then

$$x_2 = x_1 + h \quad \text{and} \quad y_2 = f(x_2) = f(x_1 + h).$$

The slope of the secant line joining P and Q is then

$$m_{PQ} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

Let's now imagine that point Q slides along the curve towards point P. As it does so, the slope of the secant line joining P and Q will more closely approximate the slope of a tangent line to the curve at P. We can in fact *define* the slope of the **tangent line** at point P as the limiting value of m_{PQ} as point Q approaches P.

As point Q approaches P, the value of $h = x_2 - x_1$ approaches zero. The slope of the tangent line at P is then

$$\boxed{m_{\tan} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}}$$

Note that we *cannot* evaluate the limit

$$\lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

by simply setting $h = 0$ since this would lead to a fraction where both the numerator and denominator are zero and we know that this is undefined.

For a particular function f, we will be able to overcome this difficulty by using algebra to factor an h from the numerator to cancel it with the h in the denominator. The resulting limit can then be evaluated by setting $h = 0$.

Example 1. Consider the function $f(x) = x^2$.

(a) Compute the slope of the secant line joining the points $(2, 4)$ and $(3, 9)$.

(b) Compute the slope of the secant line joining $(2, 4)$ and $(2+h, f(2+h))$ for $h \neq 0$.

(c) Compute the slope of the tangent line at the point $(2, 4)$.

(d) Sketch a graph of the function and the tangent line at $x = 2$.

Solution: The slope of the secant line joining the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$m_{\text{sec}} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1.1}$$

(a) Using $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (3, 9)$ gives

$$m_{\text{sec}} = \frac{9 - 4}{3 - 2} = 5.$$

(b) We use equation (1.1) with $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (2+h, f(2+h))$ to get

$$m_{\text{sec}} = \frac{f(2 + h) - 4}{(2 + h) - 2}.$$

Since $f(2+h) = (2+h)^2$, we have

$$\begin{aligned} m_{\text{sec}} &= \frac{(2+h)^2 - 4}{h} \\ &= \frac{(4 + 4h + h^2) - 4}{h} \\ &= \frac{4h + h^2}{h} \\ &= \frac{h(4+h)}{h} \\ &= 4 + h. \end{aligned}$$

(c) To find the slope of the tangent line to the graph of $f(x) = x^2$ at the point $(2, 4)$, we use

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

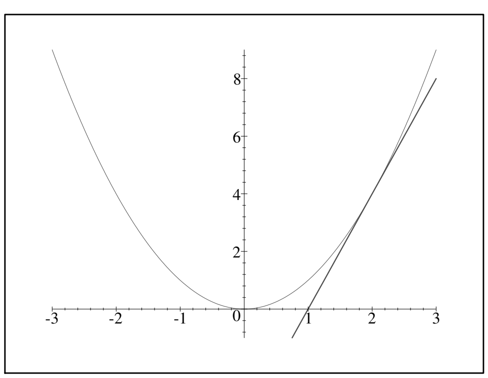
with $x_1 = 2$. Since we found that

$$\frac{f(2+h) - f(2)}{h} = 4 + h,$$

then

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} (4 + h) \\ &= 4. \end{aligned}$$

The slope of the tangent line to the curve at $x = 2$ is then equal to 4.
(d) Here is a graph of $y = x^2$ and the tangent line at $x = 2$.



$\square$

Example 2. Find the slope of a line tangent to $y = 2x^2 - 5x$ at $x = 1$.

Solution: We use

$$m_{\tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

with $x = 1$ and $f(x) = 2x^2 - 5x$. Since

$$f(1+h) = 2(1+h)^2 - 5(1+h) \quad \text{and} \quad f(1) = 2(1)^2 - 5(1) = -3,$$

then

$$\begin{aligned}
m_{\tan} &= \lim_{h \to 0} \frac{2(1+h)^2 - 5(1+h) - (-3)}{h} \\
&= \lim_{h \to 0} \frac{2(1 + 2h + h^2) - 5 - 5h + 3}{h} \\
&= \lim_{h \to 0} \frac{2 + 4h + 2h^2 - 5 - 5h + 3}{h} \\
&= \lim_{h \to 0} \frac{2h^2 - h}{h} \\
&= \lim_{h \to 0} (2h - 1) \\
&= -1.
\end{aligned}$$

The slope of the tangent line is then equal to -1. $\square$

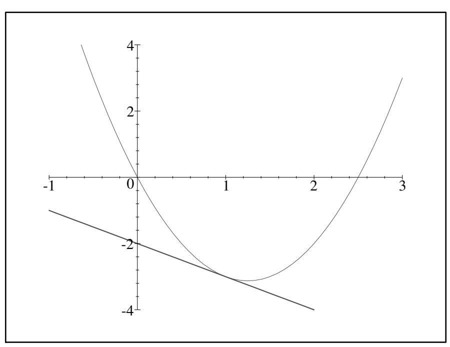


Figure 1.2: Graph $y = 2x^2 - 5x$ and the tangent line at $x = 1$.

Exercises 1.2

1. Consider the function $f(x) = 2x^2 + 1$.
   (a) Compute the slope of the secant line joining the points $(1, 3)$ and $(2, 9)$.
   (b) Compute the slope of the secant line joining the points $(1, 3)$ and $(1 + h, f(1 + h))$ for $h \neq 0$.
   (c) Compute the slope of the tangent line at the point $(1, 3)$.
   (d) Sketch a graph of the function and the tangent line at $x = 1$.
2. Find the slope of the tangent line to $y = f(x)$ at the indicated point.
   (a) $f(x) = -2x^2, \quad (3, -18)$.
   (b) $f(x) = 5x^2 - 3x + 2, \quad (2, f(2))$.
   (c) $f(x) = x^3, \quad (-2, f(-2))$. *Hint:* $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

1.3 Limits

In section 1.2, we computed limits without saying much about limits. The method we used doesn't cover all possible cases. In this section, we will look at limits in more detail.

Let us consider the following piecewise-defined function

$$f(x) = \begin{cases} 2x+1 & \text{if } x \neq 2, \\ 3 & \text{if } x = 2. \end{cases}$$

This function is made up of two parts. One part is the line $y = 2x + 1$ with the point $(2, 5)$ deleted. The other part is the single point $(2, 3)$.

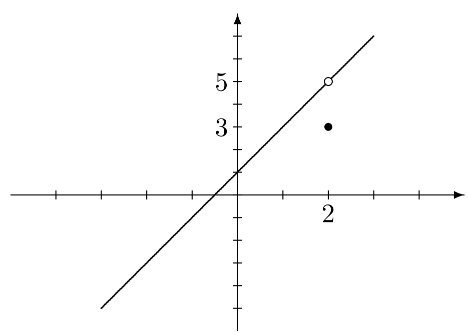


Figure 1.3: Graph of the piecewise-defined function $y = f(x)$.

As the values of x approach 2, the values of $y = f(x)$ approach 5, but $f(x)$ can never be 5 since that is where the "hole" is. The number 5 is not in the range of $f(x)$, but $f(x)$ does have values which get arbitrarily close to the value 5.

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$y = f(x)$	4.8	4.98	4.998	4.9998	3	5.0002	5.002	5.02	5.2

Table 1.1: Table of values for the piecewise-defined function $y = f(x)$.

We then say that $f(x)$ approaches the value 5 as x approaches 2, or that the limit of $f(x)$ as x goes to 2 is equal to 5.

Symbolically we write this as

$$\lim_{x \to 2} f(x) = 5.$$

Notice that it is *not* saying that $f(2) = 5$ (in fact $f(2) = 3$), it says that the values of $f(x)$ approach 5 as x approaches 2.

For a function f, the expression

$$\lim_{x \to a} f(x) = L$$

means that the values of the function $f(x)$ approach the number L as the values of x approach a from both directions. Notice that we are not interested in what happens at $x = a$.

If there is no number L that the function f approaches as x approaches a, then we say that the limit *does not exist*.

For example, consider the function $f(x) = 1/x^2$. As the values of x approach 0, the values $f(x)$ become arbitrarily large. Therefore

$$\lim_{x \to 0} \frac{1}{x^2} \text{ does not exist.}$$

For "well-behaved" functions (e.g. polynomials), we can evaluate limits by direct substitution.

Example 1. Consider the polynomial function $p(x) = 2x^3 - 5x + 1$, then by direct substitution we get

$$\lim_{x \to 2} p(x) = \lim_{x \to 2} (2x^3 - 5x + 1) = 2(2)^3 - 5(2) + 1 = 7.$$

□

Example 2. For $f(x) = \dfrac{x^2 - 2x + 1}{3x + 6}$, by direct substitution we get

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 2x + 1}{3x + 6} = \frac{3^2 - 2(3) + 1}{3(3) + 6} = \frac{4}{15}.$$

□

However, direct substitution does not always work and we will need to simplify the expression before we can substitute.

Example 3. Evaluate

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1}.$$

Solution: If we substitute $x = 1$, we get values of zero for *both* the numerator and the denominator. We eliminate this problem by factoring the numerator.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2+x-2}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)} \\
&= \lim_{x\to 1}(x+2) \\
&= 1+2 \\
&= 3.
\end{aligned}$$

□

Example 4. Evaluate

$$\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3}.$$

Solution: If we substitute $x = 3$, we get values of zero for *both* the numerator and the denominator. Rationalizing the numerator eliminates this problem.

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} &= \lim_{x\to 3} \left(\frac{\sqrt{x}-\sqrt{3}}{x-3} \cdot \frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} \right) \\
&= \lim_{x\to 3} \frac{(x-3)}{(x-3)(\sqrt{x}+\sqrt{3})} \\
&= \lim_{x\to 3} \frac{1}{\sqrt{x}+\sqrt{3}} \\
&= \frac{1}{\sqrt{3}+\sqrt{3}} \\
&= \frac{1}{2\sqrt{3}}.
\end{aligned}$$

□

Exercises 1.3

1. Consider
$$f(x) = \begin{cases} 3x+1 & \text{if } x \neq 1, \\ 2 & \text{if } x = 1. \end{cases}$$
   (a) Sketch the graph of $y = f(x)$.
   (b) Compute $\lim_{x\to 0} f(x)$.
   (c) Compute $\lim_{x\to 1} f(x)$.

2. Consider $f(x) = \dfrac{x^3-1}{x-1}$.
   (a) Complete the following table:

x	0.99	0.999	1.001	1.01
$y = f(x)$				

   (b) Guess what $\lim_{x\to 1} f(x)$ is.
   (c) Compute $\lim_{x\to 1} f(x)$ algebraically.

3. Compute the following limits.

   (a) $\lim_{x\to 3}(x^2 - 4x - 1)$.
   (b) $\lim_{x\to 2} \dfrac{x^2-5x+6}{x-2}$.
   (c) $\lim_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$.
   (d) $\lim_{x\to 0} \dfrac{3x^3-4x}{x^2+x}$.
   (e) $\lim_{x\to 1} \dfrac{x^3+3x^2-4}{x-1}$.
   (f) $\lim_{x\to 8} \dfrac{x^{2/3}-4}{x^{1/3}-2}$.
   (g) $\lim_{x\to 1} \dfrac{x^2+2x-3}{3x^2+5x-2}$.
   (h) $\lim_{h\to 0} \dfrac{\frac{1}{2+h}-\frac{1}{2}}{h}$.
   (i) $\lim_{x\to 2} \dfrac{2x^3-16}{x-2}$.

4. True or False?
   (a) $\lim_{x\to 2} \dfrac{x^2-4}{x-2} = \dfrac{0}{0}$.
   (b) If $f(x) = \dfrac{x^2-4}{x-2}$ and $g(x) = x+2$, then $f(x) = g(x)$.
   (c) If $f(x) = \dfrac{x^2-4}{x-2}$ and $g(x) = x+2$, then $\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x)$.

1.4 The Derivative

1.4.1 Definition of the Derivative

For the function $f(x) = x^2$, the slope of the tangent line at x is different for different choices of x. Indeed, the slope of the tangent line at an arbitrary point x, denoted here by $m_{\tan}(x)$, is

$$\begin{aligned} m_{\tan}(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x. \end{aligned}$$

The tangent line at any point (x, y) on the graph of $y = x^2$ has a slope equal to $2x$.

For a function f, we call the function which gives the slopes of all tangent lines, the *derivative* of f. The derivative of x^2 is then equal to $2x$.

Definition 1. The **derivative** of a function f at x, denoted by $f'(x)$, is defined by

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

provided the limit exists. If the limit exists at $x = x_0$, we say that the function f is **differentiable** at x_0.

For a function $y = f(x)$, a number of notations are commonly used for the derivative. They include

$$f', \quad f'(x), \quad y', \quad y'(x), \quad \frac{dy}{dx}, \quad \frac{df}{dx}, \quad D_x f.$$

It is a good idea to learn all these different notations.

Example 1. Compute the derivative $f'(x)$ for $f(x) = x^3$.

Solution: Since $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2.
\end{aligned}$$

Thus, for $f(x) = x^3$, $f'(0) = 0, f'(1) = 3, f'(-2) = 12$. □

Example 2. Find the derivative of $f(x) = \sqrt{x}$.

Solution: We start as in the previous example.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x} + \sqrt{x+h}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Thus, for $f(x) = \sqrt{x}$, $f'(4) = \frac{1}{4}, f'(9) = \frac{1}{6}, f'(10) = \frac{1}{2\sqrt{10}}$.

Notice that $f(x) = \sqrt{x}$ is defined at $x = 0$ ($f(0) = 0$) but the derivative *does not exist* at $x = 0$. The derivative $f'(x)$ is defined only if $x > 0$. For example $f'(-5)$ is not defined. □

Exercises 1.4.1

1. Compute the derivative of the following functions by using the definition. No shortcuts!

   (a) $f(x) = 4x^2 - 2x$. (b) $f(x) = 2x^3 + 5$. (c) $f(x) = \dfrac{1}{x}$.

   (d) $f(x) = x^4$. (e) $f(x) = \sqrt{x+1}$. (f) $f(x) = \dfrac{x}{x+1}$.

2. Given $f(x) = 1/(x-1)$, compute the derivative of $f(x)$ at $x = 5$, denoted by $f'(5)$, using each of the following methods.

   (a) *One-Step-Method:* Substitute 5 for x in the formula of the derivative, i.e., use $f(5)$ and $f(5+h)$ rather than $f(x)$ and $f(x+h)$.

   (b) *Two-Step-Method:* First determine $f'(x)$ using the definition. Then evaluate the derivative at $x = 5$.

3. On a graph of $y = f(x)$, what does $f'(2)$ represent?

1.4.2 Rules for Calculating Derivatives

We will now provide a list of rules by which derivatives can be calculated without having to use the formal definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Constant Function Rule.
For the constant function $f(x) = k$, we have $f'(x) = 0$.

Constant Multiple Rule.
For any constant k, if $f(x) = k \cdot g(x)$, then $f'(x) = k \cdot g'(x)$.

Linear Function Rule.
If $f(x) = mx + b$, then $f'(x) = m$, the slope of the line $y = mx + b$.

Sum Rule.

$$\big(f(x) + g(x)\big)' = f'(x) + g'(x).$$

In section 1.4.1 we used the definition of derivative to show that

$$\begin{array}{ll} \text{if} \quad f(x) = x^2, \text{ then} & f'(x) = 2x, \\ \text{if} \quad f(x) = x^3, \text{ then} & f'(x) = 3x^2. \end{array}$$

These two results are particular cases of the following.

Power Rule.
For any number n, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

The power rule also works if n is a fraction or a negative number. For example, if $f(x) = \sqrt{x} = x^{1/2}$, taking $n = 1/2$ we get

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

This agrees with what we found in example 2 of section 1.4.1.

Example 1. If $f(x) = 6x^5$, $f'(x) = (6 \cdot 5)x^4 = 30x^4$ by the Power Rule and the Constant Multiple Rule. □

Example 2. If $f(x) = x^7 + 6x^5$, $f'(x) = 7x^6 + 30x^4$ by the Sum Rule. □

Example 3. Compute $f'(3)$ for $f(x) = 1/x$.

Solution: Since $f(x) = 1/x = x^{-1}$, the power rule with $n = -1$ gives

$$f'(x) = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$

Thus, $f'(3) = -\frac{1}{9}$. □

Example 4. Compute $f'(2)$ for $f(x) = 2x^5 - x^4 + 5x^3 - 6x^2 - 3x + 7$.

Solution: Using all the above rules we have

$$\begin{aligned} f'(x) &= (2 \cdot 5)x^4 - 4x^3 + (5 \cdot 3)x^2 - (6 \cdot 2)x - 3 + 0 \\ &= 10x^4 - 4x^3 + 15x^2 - 12x - 3. \end{aligned}$$

Thus, $f'(2) = 10(2^4) - 4(2^3) + 5(2^2) - 12(2) - 3 = 121$. □

Exercises 1.4.2

1. Compute the derivative of the following functions.

   (a) $f(x) = 7x^5 + 5x + 1$

   (b) $p(x) = \dfrac{x^2}{9} - \dfrac{x}{3}$

   (c) $g(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

   (d) $f(x) = (3x+1)(2x-3)$

   (e) $h(x) = (2x^3+5)^2$

   (f) $V(r) = \dfrac{4}{3}\pi r^3$

   (g) $s(t) = s_0 + vt + \dfrac{1}{2}at^2$

   (h) $f(x) = (x+1)^3$

2. Evaluate the following.

   (a) $f'(2);\ f(x) = \dfrac{x^3}{6}$.

   (b) $s'(2);\ s(t) = -5t^4 + t - 1$.

   (c) $g'(1);\ g(x) = 2\sqrt{x} + 3x^2$.

   (d) $f'(-1);\ f(x) = 3/x$.

1.5 Applications of Derivatives

1.5.1 Rates of Change

Let's begin by recalling some familiar ideas about motion. Recall that

$$\text{distance} = \text{rate} \times \text{time} \quad \text{or} \quad \text{rate} = \frac{\text{distance}}{\text{time}}.$$

We wish to make this more precise.

If $s = s(t)$ corresponds to the position of an object at time t, the slope of the secant line joining two points on the curve $s = s(t)$ corresponds to the *average velocity* of the object.

If the object has position s_1 at time t_1 and s_2 at time t_2, then the **average velocity** over the time interval $\Delta t = t_2 - t_1$ is defined by

$$\bar{v} = \frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s_2 - s_1}{t_2 - t_1}.$$

As $\Delta t \to 0$, the average velocity over the time interval with endpoints t_1 and $t_2 = t_1 + \Delta t$ should provide an increasingly accurate measure of the *instantaneous velocity* at time t_1.

In fact, we define the **instantaneous velocity** at time t_1 as

$$v(t_1) = \lim_{\Delta t \to 0} \frac{s(t_1 + \Delta t) - s(t_1)}{\Delta t}$$

whenever the limit exists. This simply states that *the derivative of the position function with respect to time is the instantaneous velocity function.*

Example 1. An object moves along a straight line so that its position in meters is given by

$$s(t) = t^3 - 6t^2 + 9t$$

for all time t in seconds.

(a) Find the average velocity of the object between $t = 2$ and $t = 5$ seconds.

(b) Find the instantaneous velocity of the object as a function of time.

(c) Find the instantaneous velocity at $t = 2$ seconds.

(d) When is the object at rest?

Solution: (a) The average velocity between $t = 2$ and $t = 5$ is given by

$$\bar{v} = \frac{s(5) - s(2)}{5 - 2} = \frac{(5^3 - 6(5^2) + 9(5)) - (2^3 - 6(2^2) + 9(2))}{3} = 6.$$

The average velocity is then $\bar{v} = 6$ m/s.
(b) The instantaneous velocity of the object as a function of time is

$$v(t) = s'(t) = 3t^2 - 12t + 9.$$

(c) The instantaneous velocity at $t = 2$ seconds is then

$$v(2) = 3(2^2) - 12(2) + 9 = -3 \text{ m/s}.$$

(d) The object is at rest when its instantaneous velocity is zero. To find when the object is at rest we have to solve $v(t) = 0$. Since,

$$v(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 3)(t - 1)$$

we conclude that the object is a rest at $t = 1$ and $t = 3$ seconds. □

In general for a curve $y = f(x)$,

> The slope of the **secant line** joining two points on the curve gives the **average rate of change** of y with respect to x.
>
> The slope of the **tangent line** at a point (x_0, y_0) on the curve gives the **instantaneous rate of change** of y with respect to x at x_0.

In other words, the derivative $f'(x_0)$ represents the **instantaneous rate of change** of y with respect to x at x_0. That is true no matter what the variables x and y represent.

Example 2. A balloon in the shape of a sphere is being inflated. Find an expression for the instantaneous rate of change of the volume with respect to the radius. Evaluate this rate for a radius of 2 m.

Solution: Recall that the volume of a sphere of radius r is given by

$$V = \frac{4}{3}\pi r^3.$$

The instantaneous rate of change of the volume with respect to the radius is then

$$V'(r) = \frac{4}{3}(3\pi r^2) = 4\pi r^2.$$

For $r = 2$, we find

$$V'(2) = 4\pi(2^2) = 16\pi \approx 50.27 \text{ m}^3/\text{m}.$$

□

Exercises 1.5.1

1. The position of a car at t seconds is given by $s = 10 + 5t + 20t^2$ meters. Find the instantaneous velocity as a function of time.

2. An object is dropped from the observation deck of the CN tower so that its height in meters is given by
$$h(t) = 447 - 4.9t^2$$
where t is measured in seconds (we are neglecting air resistance.)

   (a) What is the average velocity between $t = 1$ and $t = 2$ seconds?

   (b) What is the instantaneous velocity at $t = 2$ seconds?

   (c) When will the object hit the ground?

   (d) What is the velocity of the object when it hits the ground?

3. A bushfire spreads so that after t hours, $80t - 20t^2$ acres are burning. What is the growth rate of the burning area, (the rate of change of the acreage that is burning with respect to time) when $t = 1.5$ hours?

4. A circular oil spill is increasing in size. Find the instantaneous rate of change of the area A of the spill with respect to the the radius r for $r = 100$ m.

5. Population growth (the rate of change of population size with respect to time) is proportional to the population size P. Write the latter statement as an equation involving derivative.

6. The reaction of the body to a dose of medicine can often be represented by an equation of the form
$$R = M^2 \left(\frac{C}{2} - \frac{M}{3} \right)$$
where C is a positive constant and M is the amount of medicine absorbed in the blood.

   If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

   Find $R'(M)$. (This derivative is called the sensitivity of the body to the medicine.)

1.5.2 Tangent Lines

Recall that the slope of the tangent line m_{tan} to a curve $y = f(x)$ at some point (x_1, y_1) is given by the derivative of f evaluated at x_1.

$$\boxed{m_{\text{tan}} = f'(x_1)}$$

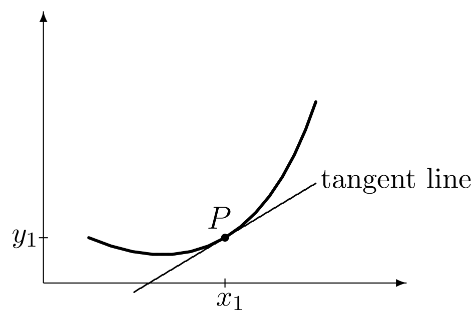


Figure 1.4: **Tangent line** at point (x_1, y_1).

Knowing the slope and the coordinates of the given point, we can find the equation of the tangent line.

Example 1. Find the equation of the tangent line to the curve $y = x^2$ at the point $(3, 9)$.

Solution: The derivative of $f(x) = x^2$ is $f'(x) = 2x$. The slope of the tangent line at $x = 3$ is then

$$m = f'(3) = 2(3) = 6.$$

If we know one point (x_1, y_1) and the slope m of a line, we can use the *point-slope* formula

$$y - y_1 = m(x - x_1)$$

to find the equation of the line.

In our case we use $m = 6$ and $(x_1, y_1) = (3, 9)$ to get

$$y - 9 = 6(x - 3) = 6x - 18.$$

The equation of the tangent line is then $y = 6x - 9$. □

Example 2. Find all points on the graph of $f(x) = 2x^3 - 3x^2 - 12x + 20$ where the tangent line is parallel to the x-axis.

Solution: A line is parallel to the x-axis if and only if its slope is zero. We then have to find all values of x for which $f'(x) = 0$.

Here we have

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1).$$

The tangent line is then parallel to the x-axis at $x = 2$ and $x = -1$. □

Exercises 1.5.2

1. Find the equation of the tangent line to $y = x^3$ at the point $(2, 8)$.

2. What is the equation of the tangent line to the graph of $f(x) = 4x - x^4$ when $x = -1$.

3. Find a point on the curve $y = x^2 + 4x - 6$ where the slope of the tangent line is 6.

4. Find all points on the curve $y = x^3$ where the tangent line is parallel to the line $y = 12x + 5$.

5. The **normal line** to a curve at a point P is defined as the line through P that is perpendicular to the tangent line at that point.

   (a) Sketch the graph of $f(x) = x^2 - 1$.

   (b) Find the slope of the tangent line to the graph at the point $(1, 0)$. Show the tangent line on your sketch.

   (c) Show the normal line to the graph at the point $(1, 0)$ and find the equation of this normal line. *Hint:* Two lines are perpendicular if the product of their slope is -1.

6. Find the equation of the normal line to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$. (see question 5.)

7. Find the value of the constant k if $y = 2x$ is a tangent line to the curve $y = x^2 + k$.

1.5.3 Graphing Polynomials

Consider the graph of the function in the picture below.

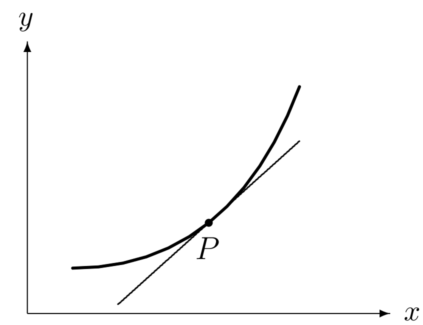


Figure 1.5: Graph of an **increasing** function with a tangent line at P.

We see that as x increases (from left to right) the y values also increase. We say that the curve is *increasing*. We also note that any tangent line to the curve will have a positive slope where the function is increasing. Since the derivative of a function determines the slope of the tangent line, we can conclude that:

> **A function is increasing on an interval if and only if its derivative is nonnegative at all points on the interval.**

A similar reasoning shows that:

> **A function is decreasing on an interval if and only if its derivative is nonpositive at all points on the interval.**

Therefore, by analyzing the sign of the derivative, we get information about where the function is increasing and where it is decreasing. This information is very useful if we want to sketch the graph of the function.

Example 1. Consider the function $f(x) = x^3 + x$. Since

$$f'(x) = 3x^2 + 1 > 0 \quad \text{for all } x,$$

the function is always increasing. $\square$

What happens at points where the derivative is zero? Three cases could occur as shown in the following pictures.

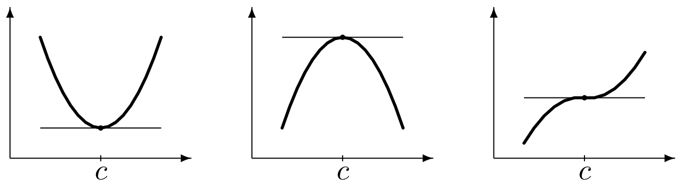


Figure 1.6: Three possible cases for $f'(c) = 0$.

The curve can have a **local minimum** (lowest point of a "valley"), a **local maximum** (highest point of a "hill"), or none of these.

For a function f, if $f'(c) = 0$, we say that the graph of f has a **critical point** at the point $(c, f(c))$. The number c is called a **critical number**.

An analysis of the sign of the derivative near a critical number will tell if it corresponds to a local maximum, local minimum, or none of these.

For a critical number c, i.e., a value where $f'(c) = 0$

- If the derivative is negative just to the left of c and positive just to the right of c, then there is a *local minimum* at $x = c$.
- If the derivative is positive just to the left of c and negative just to the right of c, then there is a *local maximum* at $x = c$.

Example 2. Consider the function $f(x) = x^3 - 12x + 5$. We have

$$f'(x) = 3(x^2 - 4) = 3(x - 2)(x + 2).$$

The critical numbers are $x = 2$ and $x = -2$. Let's look at the sign $f'(x)$.

	$(-\infty, -2)$	$(-2, 2)$	$(2, +\infty)$
$(x + 2)$	$-$	$+$	$+$
$(x - 2)$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\uparrow$	$\downarrow$	$\uparrow$

We conclude that $x = -2$ corresponds to a local maximum and $x = 2$ to a local minimum. Note that the function f is increasing ($\uparrow$) on the intervals $(-\infty, -2)$ and $(2, +\infty)$, and decreasing ($\downarrow$) on the interval $(-2, 2)$. $\square$

Guidelines for Sketching the Graph of a Polynomial.

1. Check for symmetry.
   (a) $f(x) = f(-x) \implies$ graph of f is symmetric about the y-axis.
   (b) $f(x) = -f(-x) \implies$ graph of f has symmetry through the origin.
2. Find the x and y intercepts.
3. Find all critical points, i.e., all points where the derivative is zero.
4. Construct a sign table and determine where the function is increasing and decreasing.
5. Identify all local maxima and minima.
6. Use the above information to sketch the graph.

Exercises 1.5.3

1. Follow the guideline to sketch the graph of the following functions.
   (a) $f(x) = x^3 - 3x^2 + 4$.
   (b) $f(x) = x^3(4 - x)$.
   (c) $f(x) = 3x^5 - 5x^3$.
   (d) $f(x) = x^2(2 - x^2)$.
2. Sketch the graph of $y = f(x)$ consistent with the following information. Identify all the roots, local maximum and local minimum.

	$(-\infty, -4)$	$(-4, -2)$	$(-2, 0)$	$(0, 3)$	$(3, 5)$	$(5, +\infty)$
$f(x)$	$-$	$+$	$+$	$-$	$-$	$+$
$f'(x)$	$+$	$+$	$-$	$-$	$+$	$+$

x	-5	-2	3	6
$y = f(x)$	-2	3	-3	4

3. True of False? For a polynomial f.
   (a) If $f'(c) = 0$, then f has a local maximum or minimum at $x = c$.
   (b) If f has a local maximum at $x = c$, then $f'(c) = 0$.
   (c) If f has a local minimum at $x = c$, then $f'(c) = 0$.

1.5.4 Optimization

Optimization is concerned with finding maximum or minimum of a function. The only points where a function can attain its maximum or minimum over an interval are at critical points or at the endpoints of the interval.

In the figure below, over the interval $[a, b]$, the minimum is attained at the critical number $x = c$, and the maximum at the endpoint $x = a$.

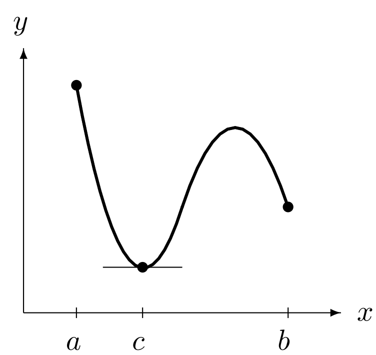


Figure 1.7: Maximum attained at $x = a$. Minimum attained at $x = c$.

Guidelines for Solving Optimization Problems

1. If possible draw a picture.
2. Locate the quantity to be optimized (let's call it Q).
3. Find an equation linking Q and another variable (say x) of the problem.
4. Find the derivative $Q'(x)$.
5. Find all critical numbers, i.e., solve $Q'(x) = 0$.
6. Test to see if the critical numbers correspond to local maxima or minima of Q by studying the signs of the derivative.
7. If the admissible values of x are restricted to a closed interval $a \leq x \leq b$, check whether the maximum or minimum value of Q is at one of the two endpoints $x = a$ or $x = b$.

Let's now look at examples that illustrate this.

Example 1. Of all pairs of positive numbers whose product is 100, which one has the smallest sum?

Solution. Let x and $100/x$ be the pair of numbers. We want to minimize their sum

$$S = x + \frac{100}{x}.$$

Differentiating gives

$$S'(x) = 1 - \frac{100}{x^2}.$$

Solving $S'(x) = 0$ gives $x^2 = 100$ which implies $x = \pm 10$. Since we want positive numbers, we only keep $x = 10$. The other number in the pair is then $100/10 = 10$.

Do we have a minimum or a maximum? The sign of the derivative around the critical number $x = 10$ will tell us.

	$(0, 10)$	$(10, +\infty)$
$S'(x) = 1 - 100/x^2$	$-$	$+$
$S(x)$	$\downarrow$	$\uparrow$

This indicates that $x = 10$ corresponds to a minimum. The pair whose product is 100 having the smallest sum is then $(10, 10)$. $\square$

Example 2. A catering service will serve a particular dinner on its menu to groups of between 20 and 50 people. For groups of size 20, the price for the meal is 12 dollars per person. For each additional person beyond 20, the price per person is reduced by 20 cents. What group size provides the service with maximum revenue?

Solution: Let $N = 20 + x$ denote the group size. Since the group size is between 20 and 50, we consider $0 \leq x \leq 30$.

Since the price per person is \$12 for $x = 0$ ($N = 20$) and is reduced by 20 cents for each additional person, then the price per person is $p = 12 - 0.2x$.

Since the revenue R corresponds to the price per person times the number of persons, we have

$$R = p \cdot N = (12 - 0.2x)(20 + x) = 240 + 8x - 0.2x^2.$$

We then have to maximize $R = R(x)$ over the interval $[0, 30]$. We find the derivative and the critical numbers.

$$R'(x) = 8 - 0.4x = 0 \implies x = 20.$$

By studying the sign of the derivative around the critical number $x = 20$, we conclude that it corresponds to a local maximum.

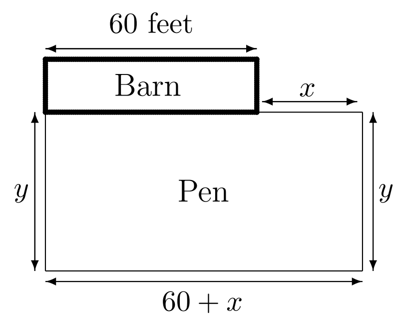
	$(-\infty, 20)$	$(20, +\infty)$
$R'(x) = 8 - 0.4x$	$+$	$-$
$R(x)$	$\uparrow$	$\downarrow$

x	$R(x)$	
0	240	
20	320	$\leftarrow$ MAX
30	300	

We then conclude that a maximum revenue of \$320 is obtained for $x = 20$, i.e., a group size of 40. $\square$

Example 3. A farmer wants to construct a rectangular pen next to a barn, that is 60 feet long, using all of the barn as part of one side of the pen. Find the dimensions of the pen with the largest area that the farmer can build using 300 feet of fencing material.

Solution: Let's draw a picture first.



The area of the pen is: $A = (60 + x)y$. To maximize the area, all fencing material should be used, therefore

$$(60 + x) + x + 2y = 300 \implies x + y = 120.$$

We then have to maximize $A(x) = (60 + x)(120 - x)$ over $0 \leq x \leq 120$. Expanding, we get

$$A(x) = (60 + x)(120 - x) = 7200 + 60x - x^2.$$

Solving $A'(x) = 0$ gives

$$A'(x) = 60 - 2x = 0 \implies x = 30.$$

Studying the sign of the derivative around the critical number $x = 30$, we conclude that $x = 30$ corresponds to a local maximum.

	$(-\infty, 30)$	$(30, +\infty)$
$A'(x) = 60 - 2x$	$+$	$-$
$A(x)$	$\uparrow$	$\downarrow$

x	$A(x)$	
0	7200	
30	8100	$\leftarrow$ MAX
120	0	

If $x = 30$, then $y = 120 - x = 90$. The dimensions of the pen that maximize the area are then $90' \times 90'$. $\square$

Exercises 1.5.4

1. The sum of the base and the height of a triangle is 20 cm. Find the dimensions for which the area is maximized.

2. Repeat example 3 above with 400 feet of fencing material available.

3. A fence is to be built to enclose a rectangular area of 5000 m^2. The fence along three sides is to be made of material that costs 5 dollars/m. The material for the fourth side costs 15 dollars/m. Find the dimensions of the rectangle that will allow the cheapest fence to be built.

4. Determine the radius and height of a cylinder with a volume of 100 cm^3 and with the smallest possible surface area (including top and bottom).

5. An egg ranch has 120 chickens, each of which produces 250 eggs per year. If fewer chicken are squeezed into the chicken coop, the resulting extra space will induce the remaining chickens to increase their egg production. Specifically, for each chicken removed, the remaining chickens will each produce 5 more eggs per year. How many chickens should the ranch accommodate in order to maximize the total yearly egg production?

6. An open-top box is to be made by cutting away congruent squares from the corners of a 12×12 sheet of cardboard. How large should the squares be to maximize the volume of the box?

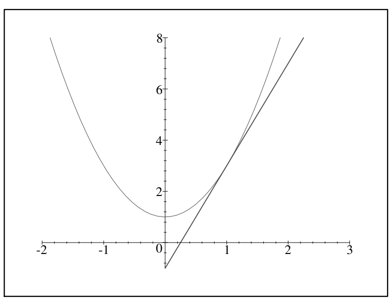
7. What are the dimensions of an open box with square base and volume 32 cm^3, that minimize the surface area of the outside?

Appendix A

Answers to All Exercises

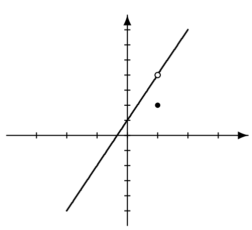
Section 1.2 (page 9)

1. (a) 6 (b) 4+2h (c) 4

(d) 

2. (a) -12 (b)17 (c) 12

Section 1.3 (page 13)

1. (a)  (b) 1 (c) 4

2. (a)

x	0.99	0.999	1.001	1.01
$y = f(x)$	2.97	2.997	3.003	3.03

(b) 3 (c) 3

3. (a) -4 (b) -1 (c) $\frac{1}{2\sqrt{2}}$ (d) -4 (e) 9 (f) 4 (g) 0 (h)$-\frac{1}{4}$

(i) 24

4. (a) False (b) False (c) True

Section 1.4.1 (page 16)

1. (a) $8x-2$ (b) $6x^2$ (c) $-\frac{1}{x^2}$ (d) $4x^3$ (e) $\frac{1}{2\sqrt{x+1}}$ (f) $\frac{1}{(x+1)^2}$
2. $f'(5)=-1/16$
3. $f'(2)$ represents the slope of the tangent line at $x=2$.

Section 1.4.2 (page 18)

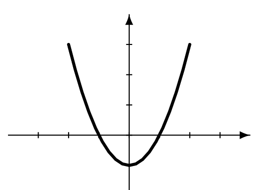
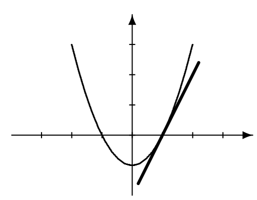
1. (a) $f'(x)=35x^4+5$ (b) $p'(x)=\frac{2}{9}x-\frac{1}{3}$ (c) $g'(x)=-\frac{1}{x^2}-\frac{2}{x^3}$
(d) $f'(x)=12x-7$ (e) $h'(x)=12x^2(2x^3+5)$ (f) $V'(r)=4\pi r^2$
(g) $s'(t)=v+at$ (h) $f'(x)=3(x+1)^2$
2. (a) 2 (b) -159 (c) 7 (d) -3

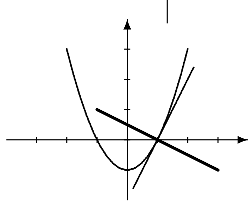
Section 1.5.1 (page 21)

1. $v=5+40t$
2. (a) -14.7 m/s (b) -19.6 m/s (c) 9.55 s (d) -93.6 m/s
3. 20 Acres/h 4. 200π m^2/m
5. $P'(t)=k\,P(t)$ 6. $R'(M)=CM-M^2$

Section 1.5.2 (page 23)

1. $y=12x-16$ 2. $y=8x+3$ 3. $(1,-1)$ 4. $(2,8)$, $(-2,-8)$

5. (a)  (b)  $m_{\tan}=2$

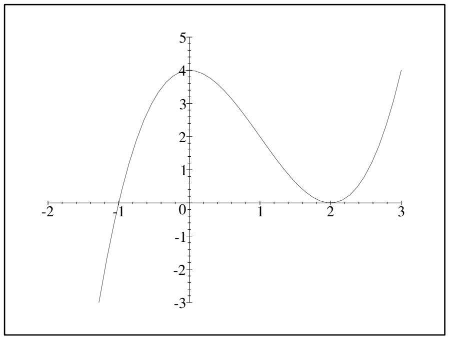
(c)  Normal line: $y=-\frac{1}{2}x+\frac{1}{2}$

6. $y=-\frac{1}{9}x+\frac{29}{9}$ 7. $k=1$

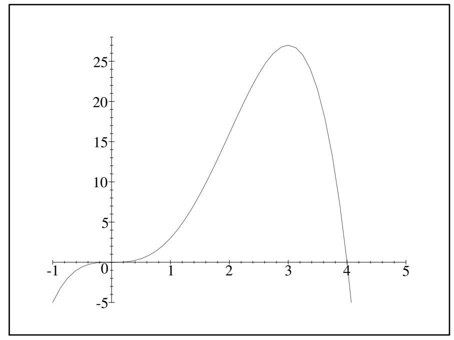
Section 1.5.3 (page 26)

1.

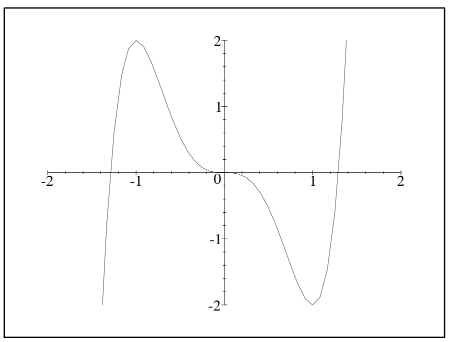
(a)



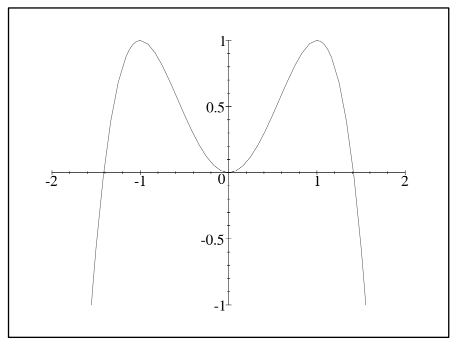
(b)



(c)



(d)



2. Any graph with roots at $x = -4, 0, 5$, local maximum at $x = -2$, local minimum at $x = 3$, increasing on $(-\infty, -2)$ and $(3, +\infty)$, decreasing on $(-2, 3)$, and passing through the given points is a correct answer.

3. (a) False (b) True (c) True

Section 1.5.4 (page 30)

1. Base and height are 10 cm
2. $x = 55$, $y = 115$. Dimensions: $115' \times 115'$
3. 50 m × 100 m 4. $r = \sqrt[3]{50/\pi}$, $h = 100/(\pi r^2)$
5. 85 chickens 6. Squares are 2×2
7. The box is 4 cm × 4 cm × 2 cm